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UNIVERSITY OF ALBERTA

ESSAYS INVESTIGATING FINANCIAL THEORY AND PRACTICE

BY

© PRADEEP JALAN

A THESIS
SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

IN

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Pradeep Jalan
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G. Barone-Adesi ....
supervisor
L. Kryzanowski ....
B. Dahlby BW Dahlby
R. Korkie ....
P. Talwar P P Talwar
Y.S. Kim Youngsoo Kim

Date: March 8, 1990

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ESSAYS INVESTIGATING FINANCIAL THEORY AND PRACTICE

ABSTRACT

The thesis comprises three papers, which are described individually in the following paragraphs.

CALLS ON CONVERTIBLES: ex ante optimality in a cooperative context. The paper re-examines the factors involved in the formulation of an optimal call policy by the issuers of convertible callable debt securities. The focus of the paper is on an ex ante optimal policy at the time of floating the new issue. The existence of differential tax treatment of debt and common stock, together with perceived investor risk aversion, serve to provide sufficient reason for convertible debt issuance and for observed call policy. When the need to seek corporate financing on an ongoing basis is recognized, the rationale for observed corporate call policy is reinforced.

UTILITY BASED ASSET PRICING: a test of CAPM and Beta sensitivity. This paper looks at the traditional CAPM model from a Utility Based Asset Pricing (UAPM) framework. This framework enables one to broaden the traditional measure of risk, Beta, to a more generalized measure of risk. The linear risk-return relationship is retained, with the generalized Beta replacing the more conventional Beta. This formulation permits nested tests of the CAPM within an equilibrium utility based pricing relationship. The model also allows for the examination of the sensitivity of asset risk premiums required at market equilibrium, to changes in aggregate Relative Risk Aversion.

The results indicate that UAPM fails to dominate CAPM, for the data set used in this paper.

RELATIVE RISK AVERSION ESTIMATION. This paper uses stock market data to obtain estimates of aggregate constant relative risk aversion for investors with preferences characterized by isoelastic power utility. The earlier work of Brown & Gibbons (1985) is extended by using multivariate methodology.

The results indicate that the model has low explanatory power and the estimates may be subject to small sample bias. The study fails to remedy the current absence of a robust model for risk aversion estimation. Further research is required.

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CALLS ON CONVERTIBLES

ex ante optimality in a cooperative context

There is an ongoing debate in the finance literature on why firms delay calling convertible debt issued by them, when it seems optimal to call as soon as the debt has a market value that exceeds the call price. A number of reasons have been put forth to explain the observed behavior, but the explanations do not seem strong enough to justify the prevalence and extent of this theoretically suboptimal practice. This paper takes a fresh look at the problem and tries to find a plausible rationale for observed call behavior on convertible bonds. The main difference in approach from prior work on this area is the emphasis on the factors that lead to the successful issuance of new convertible debt. This analysis leads us to the conclusion that there exist factors which implicitly inhibit firms from exercising their right to call at the first available opportunity.

Since the work of Ingersoll (1977a) and Brennan & Schwartz (1977) it has been recognized that given the assumptions under which the Modigliani-Miller (1958) theorem on the irrelevance of corporate capital structure are applicable, it is clearly in the best interests of the firm's existing shareholders to call outstanding convertible debt issues whenever the market value of the debt exceeds the call value. But Ingersoll (1977b) provides empirical evidence to show that the majority of firms do not seem to follow this policy. A variety of rationales have been put forward to explain the observed failure of firms to practice this policy. Most of the arguments that are presented are linked to the observed negative stock price reaction when a convertible bond issue is called. The explanations offered include the following:

1. Information effects of forced conversion.

In the absence of complete information, market participants may infer that the need to force conversion implies that the firm (which has superior information) anticipates a reduction in the firm's future cash flows, which translates to lower earnings and consequent fall in value of the firm. Therefore firms delay calling to avoid sending a negative signal to the marketplace. Harris and Raviv (1985) set up a signalling model which demonstrates this "bad news" effect and the motivation for delaying forced conversion. Acharya (1988) tests the implication of this model and fails to reject the model. In support of this hypothesis, Ofer and Natarajan (1987) find that firms which call convertible bonds exhibit a decline in their performance subsequent to the call.

2. Managerial incentives to maximize earnings per share.

The compensation agreements of managers may cause an incentive incompatibility problem, putting their interests at odds with those of the shareholders. If earnings per share is one of the criteria used to judge managers' performance, and the size of their bonus pay is linked to performance, then managers have an incentive to delay calling because a call would increase the outstanding shares and reduce the earnings per share.[1]

3. Effect of reduced leverage.

This explanation is in line with observed behavior that the firm's stock price declines when a firm reduces its leverage. Forcing conversion of callable convertible debt, which decreases leverage, depresses the stock price. Therefore firms have an incentive to delay calls on bonds. The differential tax treatment of bonds vis-a-vis equity and information asymmetry have been offered as potential reasons for this effect. Mikkelson (1981,1985) finds some evidence to support this line of reasoning, but is unable to isolate the relative impact of the two contributory factors.

4. Wealth redistribution effects.

Changes in the risk attributes of cash flow streams available to specific category of investments in a firm alters their market value, even though the total value of the firm is unchanged. A call induced conversion from debt to equity may imply that outstanding debt becomes less risky, thereby increasing their market valuation. This results in a shift of wealth from equity to debt. Therefore, firms may delay forced conversion. Masulis (1980, 1983) documents evidence confirming the existence of this effect.

This paper posits that such explanations have limited impact on the firm's decision to defer forced conversion of convertible debt. The Ingersoll-Brennan-Schwartz argument may be valid in a situation where the firm's cash flows are distributed to only two types of claimants. But the reality is that there are three major claimants to the firm's cash flows. Besides the debt and equity holders, the government has a substantial claim. Thus the situation is best analyzed not as a simple two person zero sum game, with the bondholders' gain being the shareholders' loss, but as a more complex three person game. Though tax effects have been mentioned as a potential reason for deferring forced conversion, the analysis has been limited to loss of tax shields subsequent to the call date. The model analyzed in this paper takes an ex ante look at the tax implications of issuing convertible callable debt as a means of delayed equity financing. This type of analysis allows for co-operative behavior amongst these two categories of securityholders, as it is in their best interests to try and reduce the government's share of corporate cash flows. This increases the equity value of the firm, which increases the expected future payoffs to *both* the existing equityholders and the convertible bondholders. The nature of contracts agreed to by the different classes of security holders, and the demands of a competitive marketplace then serve to efficiently allocate the residual cash flows amongst them.

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[1] Jensen and Meckling (1976) were the first to examine agency conflicts in a financial setting.

The rest of the paper is organized as follows. Section I reviews the major developments in corporate capital structure theory which are relevant to this paper. Section II looks at the implication of the original Modigliani-Miller (1958) assumptions for convertible bond issues. Section III examines the effect of market imperfections and taxes and provides a rationale for implicit contract behavior between existing equityholders and new investors.

Section IV analyzes the ex ante conditions required for a co-operative coalition to form. It also identifies the inducement to cheat by equityholders and the power of bond markets to enforce the implicit contract. Section V concludes.

I. REVIEW OF RELEVANT CAPITAL STRUCTURE THEORY

Finance theory has yet to resolve "the capital structure puzzle" (Myers, 1984). A base case scenario was developed by Modigliani-Miller in 1958, who established that capital structure is irrelevant if there are no taxes and perfect markets exist. The implication of corporate taxes and their differential impact on debt and equity cash flows gave rise to the corrected Modigliani-Miller (1963) model, which explicitly recognized the corporate tax shield advantage of debt. In the absence of countervailing factors, this meant that firms would optimally choose only debt to fund their financial needs. The unbridled demand for debt is offset by a number of factors limiting the availability of debt finance. Modigliani (1982) lists the major ones as: (a) bankruptcy costs; (b) agency costs; (c) moral hazard; (d) declining incremental value of the debt tax shield as debt is increased. Therefore there is some justification for the existence of an interior optimal debt to equity ratio. However, finance theory has not yet rationalized the reasons for the vast range and array of capital structures in existence.

This paper finds a tax-based rationale for the issuance of convertible callable debt and the observed call policy of the firm.

Though an argument can be made for convertible bond issuance as a means of resolving some aspects of agency costs and the moral hazard problem, amongst others, detailed discussion is restricted to the tax aspect.

Static Theories on the Value of Debt Tax Shield

Miller (1977) re-opened the debate on the tax advantage of debt by claiming that the debt tax subsidy vanishes once personal taxes, which discriminate against debt vis-a-vis equity, are considered. In effect, Miller argued that the personal tax rates of the marginal individual investor is such that the double taxation of equity income at the corporate and personal level is exactly offset by the personal tax on debt income, thereby making debtholding irrelevant to firm value. DeAngelo and Masulis (1980) modelled the effects of differential personal taxes together with the presence of non-debt tax shields such as depreciation and tax credits. Their analysis overturns Miller's result, as once again debt tax shield is shown to be a relevant parameter when optimizing firm value. In addition, DeAngelo and Masulis recognize the relevancy of leverage to individual firms when there is cross-sectional variation in corporate tax rates, since the threshold values of personal tax rates required for no net tax shield effects would depend on the marginal corporate tax rate and create corporate capital structure clienteles. Modigliani (1982) expresses skepticism for Miller's (1977) conclusions and points out the tendency for unstable corner solutions in Miller's framework. Modigliani shows that leverage is valuable in a mean-variance portfolio optimization framework which allows for a known, constant rate of inflation.

Despite the wealth of analysis cited above, the debate on the value of debt tax shields rages on. Although the exact marginal value of debt tax shields may be an open issue, the weight of evidence seems to favor the presence of debt tax shields. For the purposes of the model presented in this paper, all that is required is that there is some market value attached to the corporate tax subsidy on debt. Though the model in this paper could explicitly incorporate personal tax effects, the basic thrust of the argument would remain unchanged. Since there are no obvious advantages to offset the added complexity, the distortional effects of personal taxes are not explicitly considered.

Dynamic Pecking Order Theory of Capital Financing

Myers (1984) coins the term "pecking order" to describe a popular theory on corporate financing decisions. In contrast to the static tradeoff framework of the works cited above, this theory hypothesizes that firms finance their capital requirements by following a ranked menu of choices. Internal financing tops the list. If external funds have to be tapped, firms issue debt, followed by hybrids such as convertible bonds. Equity is issued as a last resort. In a dynamic context, firms choose the best option available to them at the time. Therefore, no optimal capital structure mix exists. In fact, internal equity is considered most desirable, while external equity is considered least desirable! Observed capital structures indicate nothing more than the cumulative choices made by a firm over time. Myers and Majluf (1984) model some aspects of the pecking order theory. Their model predicts that given informational asymmetry, if certain conditions hold, firms will have an incentive to maintain financial slack in the form of cash and marketable securities, and preferentially use internally generated funds. If external funding is necessary, debt is preferred to equity.

The tax-based arguments presented in this paper support the pecking order theory *if* external financing is required. Therefore, the model presented in this paper can accommodate aspects of both types of theories on corporate structure. Until a more detailed theory which incorporates both static and dynamic aspects of corporate financing decisions is available, compatibility with both theories is the best that can be achieved. The model developed below can be considered a first attempt at a partial equilibrium analysis of the factors conducive to issuing a convertible bond when the tax subsidy of debt is the prime motivation. Other potentially valid reasons for issuing a convertible bond are abstracted away. This (admittedly restricted) model serves to provide a rationale for observed convertible bond call policy, when viewed in an ex ante context.

II. TAX-FREE & FRICTIONLESS ENVIRONMENT

Consider a utopian world where there are no taxes. This precludes the use of complicated (or simple, for that matter !) tax strategies designed specifically to reduce the amount of taxes paid by market participants. Assume further that competitive markets exist which allow for frictionless, atomistic, and instantaneous market clearing. In this most desirable of worlds the classical assumptions of Modigliani-Miller (1958) apply. In this setting let us examine the options available to a corporation requiring an infusion of funds for the purpose of carrying on or expanding its activities.

It is assumed that the risk characteristics of the firm's operating cash flows are unaffected by the financing. Prior to the financing, the firm has a simple capital structure comprising of straight debt and common equity. Though other options for obtaining funds are certainly available, we restrict our attention to three classes of securities; namely equity, straight debt, and convertible callable bonds. It is possible to examine individually the impact that floating these securities will have on firm value, and on the value of specific classes of existing security.[2]

1. **All Equity :** In this case there is no effect on the firm's value, except for the addition of the market value of fresh equity. The increased equity decreases the firm's leverage and all other things being equal, enhances marginally the value of the bondholders' claims due to the reduced chances for bankruptcy. This increase will be at the expense of existing shareholders, since the new equity will be priced to reflect the new capital structure. If the fresh equity is not offered at a discount to market price, old and new stockholders are not subject to any differential impact, and all the shares have a proportional claim to the residual value of the firm.[3]

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[2] Though a change in financial structure will have implications for the riskiness of individual securities, these shifts in riskiness are assumed to be of secondary importance in order not to detract from the main focus of this paper.

[3] Though the equity issue is of necessity sold at a discount to induce investor participation, these and other costs of flotation are ignored for the moment under the frictionless markets assumption.

2. All Straight Debt : In this situation once again the value of the firm remains intact, except for the fresh capitalization. The effects of increased leverage imply that the equityholders stand to benefit marginally at the expense of existing bondholders who are exposed to higher risk of default. Note, however, that in an efficiently functioning market the new bond issue would be fairly priced to yield market rates commensurate with the perceived increase in risk.

3. All Convertible Callable Bonds : This type of security is a hybrid of debt and equity, the conversion privilege allowing for a costless switch from debt to equity. Therefore, it has substantially higher capital gains potential than straight debt. Typically, the offset for this desirable feature vis-a-vis straight debt is a lower coupon rate. Once again, in an efficient market setting, the security would be priced to yield market clearing rates, without upsetting the basis for firm valuation. Without the call feature, the appropriate market clearing rate could be readily determined. However, given that the management of the firm can call the issue, the market has to have some ex ante expectation of the firm's call policy. If, as argued by Ingersoll-Brennan-Schwartz, the firm's policy is to call as soon as the market price of the convertible exceeds its call value, and the market realizes this to be the case, the value of the convertible bond would be substantially below what it would fetch if no forced conversion occurs. This is because the convertible would be shorn off its most attractive feature - that of enhanced potential for capital gain, when compared to straight debt. In fact, under these circumstances straight debt with no call provisions may actually have better capital gains potential, if a favorable shift in the interest rate structure occurs.

Given these circumstances, convertibles which pay lower coupons than competitive bond yields, would tend to be dominated by securities available in the bond markets, and the issue would be impossible to sell except at a discount to face value. For market clearing to occur at face value, the coupons would need to be raised substantially, or the conversion premiums reduced drastically at the time of issue. This would make the convertible callable bond similar (if not identical) to either debt or equity. The only way it is possible to retain the essential features of a viable convertible bond, and be able to successfully market the issue, is for the firm to follow a call policy which does not do serious damage to the convertible holders' position. This implicitly imposes some constraints on the kind of policy a firm can follow.

From the standpoint of the buyer, given that ideal competitive markets exist for securities, it is necessary that the firm signal in a credible fashion that it would not induce forced conversion under circumstances detrimental to the interests of bondholders. To the extent that there are specific covenants specifying the conditions under which a call may be undertaken, there are some legally binding safeguards which protect the convertible bondholders' interests. In most cases, there are specific provisions which ensure that no call is possible for a specified period from the date of initial offering. However, the contractual obligations do not go far enough. There is, therefore, a need for implicit (i.e. not legally binding) contracts which are credible and accepted by all market agents.

This paper argues that it is this implicit understanding which is at the root of the firm's perceived ex post "suboptimal" behavior. The firm knows that its existing stockholders would benefit from forced conversion. It could renege on any implicit understanding that a call policy which causes serious damage to bondholder's interests would not be pursued. However, this is a viable policy to follow only if the firm needs to capitalize just once in its existence. Moreover, this potential for moral hazard is recognized by the marketplace. If buyers have no means at their disposal to enforce the implicit contract it would not be possible for firms to issue callable convertibles, as buyers would rightfully shy away from purchasing securities which would be perceived as redundant from their standpoint.[4] Therefore, there is a need for the issuer to be credible to the buying public about its intention not to call at the first available opportunity.

This credibility is achieved by the firm's reputation, and prior policy regarding its capital market operations. This implies that the marketplace recognizes that the firm needs to make repeated use of the capital markets. This allows the marketplace to police the firm's actions and enforce any implicit contract by threatening to impose harsh punitive measures or boycott any future issue the firm may need to make. An empirical implication of this argument is that mostly firms with a stable longstanding reputation would be able to successfully tap the callable convertible bond market. Less reputable firms would be constrained to issuing either non-callable convertibles, or callable convertibles with more stringent covenants specifically designed to restrict the firm's ex post call strategy.

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[4] Since they would then assume that the firm would call the bond as soon as it was optimal to force conversion, depriving them of any potential for capital gain. Under these circumstances the security would offer payoffs no better than an ordinary bond and would be discounted to yield comparable returns.

To this point, the discussion has addressed the need for an implicit contract to safeguard the interests of the buyer of a callable bond. There is a need to find a convincing motive for the buyer to opt for this risky implicit contract. Given that a well functioning options market exists in our utopian world, it is easy to extend the original Modigliani-Miller (1958) arguments to show that any individual could in effect create his own convertible bond, without having to rely on the reputation of the issuer to inhibit a forced call which is detrimental to the interest of boldholders. There is also a need to justify why the firm would resort to such a complex form of financing when it does not result in any increase in the firm's value. In our utopian world, we are indeed hard pressed to find a plausible reason for market participants' engaging in admittedly risky implicit contractual behavior. This paper posits that the reason for this behavior is to be found in the imperfect world in which these securities are issued.

III. THE EFFECT OF MARKET IMPERFECTIONS AND TAXES

In the absence of taxes and market imperfections, it is clear that there is no real economic benefit to either the issuer or the holder, in undertaking the risky and complex transactions involved in callable convertible bond financing. The presence of these factors alters the analysis in the following fashion.

Market imperfections : Though options markets exist, they do not have the depth or breadth to cater to all the needs of investors. Traded call and put options do not have the extended time to maturity or the dual sensitivity to underlying stock price and the term structure of interest rates which are characteristic of convertible bonds. Technically, it is feasible to put together combinations of securities which would synthesize the convertible bond. In practice, however, even if such a combination could be created, the mechanics of doing so may require the incurring of prohibitively high transaction costs. Therefore, market friction and incompleteness imply that risk averse individuals may be able to improve their optimal investment allocation if their choice set is expanded to include callable convertible bonds. This makes it feasible for issuers' of these securities to extract monopolistic rents from investors, thereby making both parties to the transaction better off.[5]

Corporate taxes : The introduction of taxes serves to complicate the environment in which the corporation functions. This is because corporate taxes create a third type of claimant to the net cash flows of the firm, namely the government. This claimant is very different from the debt and equity participants, because of the passive nature of it's involvement. The government participates in absentia, it's share of the income pie fully specified and regulated by legislative fiat.

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[5] The presence of traded issues which are close substitutes to a contemplated new offering places an upper bound on the rents a firm can claim successfully.

The presence of this third party is certainly not welcomed by the other claimants to the firm's cash flows. Whereas bondholders and equityholders would be natural adversaries if left alone, the unwelcome presence of the government serves to forge an uneasy alliance between the two. Despite their mutual distrust, both recognize the negative effects of corporate taxation, which bleeds away a portion of the firm's income, on their claim to the firm. Strangely enough, corporate taxation serves to cement the relationship between the bondholders and equityholders by providing a strong incentive to equityholders, whose interests management serves, to use debt financing as a preferred source of capital funding.

This is because of the differential tax treatment of the cost of financing via debt and equity. Payment of interest to debtholders is considered a legitimate business expense and therefore paid out of pretax income. Stockholders, on the other hand, have residual claim on the firm's cash flows after interest *and* tax is paid. If the market functions in an efficient manner and investors seek a fair return on debt and equity which is consistent with their risk characteristics, there is an asymmetry built into the way the issuers and the holders of debt see the fixed income market. This is because the firm in effect pays only a part of the cash flows to the bondholders, the balance being funded by the government in terms of tax savings from using debt rather than equity. As long as the firm does not have to pass on all the tax savings to the buyers of the debt issue, it stands to increase its equity value by using debt financing when compared with equity financing.[6]

The foregoing analysis provides a strong incentive to firms for choosing debt financing, all other things being equal. If the firm is a going concern with financing requirements of an ongoing nature, there is a need to tap the debt markets regularly. This recurring need to use debt financing by the firm provides a further rationale for bondholders ability to enforce any implicit contracts, since both sides recognize that the firm cannot forsake debt markets without significantly increasing its tax adjusted financing cost in the future.

The justification for debt financing by the firm is straightforward enough, given corporate taxation. It is not yet obvious as to why a firm would resort to issuing convertible debt securities. The rationale for this is a little more subtle.

The reason for the firm resorting to convertible debt as a means of financing has its roots in the need for firms to raise funds through equity in addition to debt. This is because unbridled resort to debt financing can create a highly leveraged firm which is vulnerable to bankruptcy. A direct consequence of too high a level of debt is a significant increase in agency costs due to more stringent monitoring by creditors. The potential threat to a firm's viability is a source of great concern to debtholders, because the presence of limited liability restricts them to securing their loans with just the assets of the firm. Therefore, prudent lenders will not extend loans to firms which are highly leveraged. The recent increase in the size of the junk bond market would still allow the firm to obtain debt financing, but only at a higher cost and on terms which are highly restrictive. A component of the enhanced cost of debt financing is the increase in monitoring costs. Firms which are easy to monitor may not be penalized excessively, but some firms may find the burden this cost imposes to be in excess of the projected tax subsidy. In addition to these investor demand considerations, the presence of non debt tax shields imply an upper bound to the supply of tax induced debt financing by corporations. Therefore, it is unrealistic to expect firms to resort only to debt funding. Equity capital must also be raised in order to provide some cushion to firms' ability to face adverse circumstances.

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[6] Whether this condition holds in practice is an empirical issue. In a recent paper, Alderson et al (1987) have shown that the average yield for dutch auction rate preferreds (DARPS) when compared with average commercial paper yields indicate a rough 2:1 split (firm: investor) of the tax subsidy available to corporate investors (who can claim as deductions a large part of dividend income) of DARPS. This suggests the potential for co-operative behavior between issuers and investors of convertible callable bonds.

Firms can boost their equity by a number of methods. The first is retaining a portion of their net earnings, the rest being paid out as dividends. This is in fact the method most commonly used to fund the ongoing financial requirements of firms. A major advantage of this method is the absence of any flotation costs. But it is not always sufficient to meet the need for larger capital infusions from time to time, occasioned by the natural growth of healthy firms. Another method is a rights issue, which allows existing shareholders to retain their share of a firm's residual cash flows if they do not sell their rights to third parties. A third method is to tap the equity markets for additional equity funding. However, none of these methods is as effective as issuing convertible bonds, which can be viewed as delayed equity financing.[7]

Convertible bonds are generally sold at a premium to their conversion value, therefore the same amount of financing can be obtained at a lower dilution of existing equity, when conversion does in fact occur. Firms are able to extract this premium because the dual nature of the convertible security provides a valuable option to the owners of the convertible. Ingersoll (1977a) has shown that a convertible bond can be considered a combination of a straight bond and a warrant. On the other hand, when the convertible bond is floated primarily as a means of deferred equity financing, it can also be analyzed as a combination of stock and a put option.[8] This type of financing allows firms to take advantage of the tax deductibility of the coupon payments, till conversion occurs. This tax shield effect enhances the value of firms, as tax outflows are reduced. Because this tax shield is not available to any other method of equity financing, convertible bonds are the most effective form of raising equity capital, albeit after some time delay.[9]

The differential tax treatment of interest and dividend payouts, together with the need for periodic equity financing has provided a rationale for firms to issue convertible debt. The arguments in section II showed that credible implicit contractual behavior by firms is necessary to induce investor participation. Investors in convertible bonds recognize the need for co-operative behavior on the part of firms, and are willing to participate in the co-operative game if they can enhance their risk adjusted return by extracting a share of the enhanced firm value due to the tax subsidy created by interest deductibility. Given that there is sufficient motive for the existing equityholders and the convertible bondholders to engage in co-operative behavior, it is meaningful to analyze in some detail the conditions under which the prospective players will choose to participate. The next section seeks to identity the bounds within which it is economically feasible to form viable coalitions between existing equityholders and convertible debtholders.

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[7] Especially in the context of the discussion, which is optimal induced conversion.

[8] At the time of issue the put option is in-the-money with exercise price equal to the conversion price.

[9] In addition, there is no need to offer a discount to fair market price, as would be required for an equity issue. Flotation costs are also generally lower in the bond market vis-a-vis the equity market.

IV. CO-OPERATIVE GAME PLAYING BEHAVIOR

One of the reasons for a firm to float a primary issue of convertible bonds is that it is the most effective means of equity financing.[10] Therefore the primary concern from the perspective of the existing equityholders, is that the convertible bond issue does not harm their interests vis-a-vis an alternative equity issue. Potential investors also desire to maximize the expected risk adjusted return on their portfolio, subject to their risk tolerance and other relevant investment constraints. Some of these investors may prefer the convertible bond to the equity as an investment for their portfolio, due to the differential cash flow and risk characteristics of the two securities. Therefore a new convertible bond issue will find buyers at par value if it is fairly priced relative to the common equity, on a risk adjusted basis.[11] The tax deductibility of debt coupon payments makes it feasible for both classes of investors to enhance their expected risk adjusted payoffs under convertible debt financing, at the expense of the tax authorities. Therefore, the two classes of securityholders will form a coalition in order to share the cash flow sheltered from corporate taxation.

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[10] Firms may have other reasons to issue convertible bonds. In the context of optimal call policy to induce forced conversion however, these are of minor significance and are therefore ignored.

A detailed examination of the conditions for successful issuance of the convertible bond issue requires comparing the incremental payoffs available to existing equityholders and investors when convertible debt is issued as an alternative to common shares. It is assumed that ex ante, the firm and market participants hold the common belief that all convertible bondholders would voluntarily convert to common equity on the last possible date it is feasible for them to do so. Therefore the relevant time frame for analysis is from the date of issue to the last date when voluntary conversion can take place, as the two classes have identical cash flow implications once conversion takes place. It is also assumed that this date coincides with the last date for exercising the call privilege by the firm, and that ex ante the firm is not expected to exercise the call privilege without the tacit approval of bondholders. The following notation will be used for the subsequent analysis:

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[11] If not, market clearing will require a discount on the face value. It is not unusual for new issues to be slightly underpriced relative to ex post equilibrium value. This is an intangible (though not insignificant) part of the cost of floating a new issue. For ease of analysis, it is henceforth assumed that the coupon rate is adjusted prior to the date of issue to ensure that the new issue clears the market at par value.

V_E = Market value of existing common equity [12]

V_D = Market value of all other existing securities outstanding

V_F = Fresh capital raised by new issue

V_{EF} = $V_E + V_F$

Δ_E = Ratio of new shares to old shares under equity financing

DIV_E = Present value of expected dividend payments for new equity issue (till last date for conversion)

Δ_C = Ratio of new shares to old shares under convertible bond financing (assuming 100% future conversion)

INT_C = Present value of convertible bond coupon payments (till last date for conversion)

TS_C = Present value of expected tax shields from convertible coupon payments (till last date for conversion).

PUT_C = Market value of put feature at time of issue.

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[12] For ease of notation, it is assumed that the existing equity is priced by the market on the basis that all future capital needs are financed by common equity. A different market prior will not materially alter the analysis.

A. Conditions for co-operative behavior

Under the assumption that there are no wealth redistribution effects (due to shifts in risk) on existing securities, the value, V_D, of existing non-equity securities does not change. Therefore the total present value of expected future payoffs to the coalition of existing equityholders and new issueholders is given by V_{EF} under equity financing. For convertible bond financing, the projected combined payoff to the coalition is enhanced by the present value of expected tax savings, TS_C, due to coupon deductibility.

The two classes of securities are identical in the case of equity financing. Therefore the allocation of the coalition value, V_{EF}, is proportional to their numerical ratio, Δ_E. Under convertible debt financing the total value of the coalition is $\{ V_{EF} + TS_C \}$. However, the cash flow streams available to the two classes are not the same until conversion occurs. The incremental cash flows available to new investors is given by $\{ INT_C - DIV_E \}$. In addition, the class of convertible bondholders is provided with a valuable put option, PUT_C. The *quid pro quo* is the smaller proportion of the firm they control on full conversion, as Δ_C is generally smaller than Δ_E.[13] Therefore the market capitalization for the two classes will deviate from strict proportionality of (fully diluted) equity holdings, Δ_C.

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[13] The fact that the new security is effectively disenfranchised till conversion occurs has important corporate control implications. Additionally, convertible bonds may be effective in reducing the potential for moral hazard and adverse selection when informational asymmetries exist. These important issues offer additional reasons, besides differential tax treatment, for preferring convertible debt to equity as a means of equity financing.

The expected payoffs to the holders of existing equity and the new issue, under equity financing and convertible bond financing (assuming full conversion), are listed in table 1.

Table 1

Allocation of market capitalization of the coalition value between old equityholders and investors in the new issue under convertible bond and equity financing.

	MARKET VALUATION	
SECURITY CLASS	**Convertible financing**	**Equity financing**
Old Equity	$[\ 1/(1+\Delta_C)\ ]\ [\ V_{EF} + TS_C - (INT_C - DIV_E) - PUT_C\ ]$	$[\ 1/(1+\Delta_E)\ ]\ V_{EF}$
New Issue	$[\ \Delta_C/(1+\Delta_C)\ ]\ [V_{EF} + TS_C - (INT_C - DIV_E) - PUT_C\ ] + \{\ (INT_C - DIV_E) + PUT_C \}$	$[\ \Delta_E\ /(1+\Delta_E)\ ]\ V_{EF}$
Coalition	$V_{EF} + TS_C$	V_{EF}

If the market is efficient, and no information asymmetry exists, the above valuations are fully known to all market participants. Old equityholders and new investors have an incentive to form a coalition if ex ante the two classes of securityholders are *both* better off under convertible debt financing. Therefore co-operative game playing requires the following propositions to hold:

Proposition I

Existing holders of common equity will be willing to form a co-operative coalition with convertible bondholders if and only if market conditions are such that *ex ante*

$$[\ 1/(1+\Delta_C)\ ]\ [\ V_{EF} + TS_C - (INT_C - DIV_E) - PUT_C\ ] \ > \ [\ 1/(1+\Delta_E)\ ]\ V_{EF}$$

Rearranging terms yields the following condition

$$[\ 1/(1+\Delta_C)\ ]\ TS_C + [\ 1/(1+\Delta_C) - 1/(1+\Delta_E)\ ]\ V_{EF} > [\ 1/(1+\Delta_C)\ ]\ [\ (INT_C - DIV_E) + PUT_C\ ]$$

This condition can be interpreted to mean that old equityholders have an incentive to engage in cooperative behavior if *ex ante* the benefits to them of the tax subsidy and smaller dilution due to a convertible debt financing is higher than the cost of providing a larger cash outflow and put protection to the convertible bondholders.

Proposition II

New investors will be willing to subscribe to a convertible bond issue and form a co-operative coalition with old equityholders if and only if market conditions are such that *ex ante*

$$[\ \Delta_C /(1+\Delta_C)\ ]\ [\ V_{EF} + TS_C - (INT_C - DIV_E) - PUT_C\ ] + [\ (INT_C - DIV_E) + PUT_C\ ] \quad > \quad [\ \Delta_E /(1+\Delta_E)\ ]\ V_{EF}$$

Rearranging terms (and simplifying) gives the following condition

$$[\ \Delta_C /(1+\Delta_C)\ ]\ TS_C + [\ 1 /(1+\Delta_C)\ ]\ [\ (INT_C - DIV_E) + PUT_C\ ] \quad > \quad [\ \Delta_E /(1+\Delta_E) - \Delta_C /(1+\Delta_C)\ ]\ V_{EF}$$

This condition states that on balance investors will have an incentive to subscribe to a convertible bond issue if their share of the tax subsidy together with the superior cash distribution and put protection is larger than the value of foregone equity in the firm, when compared to a direct equity investment.[14]

The two conditions above define the pricing bounds within which the co-operative game is played. It is clear that the surplus that is extracted by the coalition of existing equityholders and convertible bondholders is the present value of the expected tax shields due to interest deductibility. The loser in this three person co-operative game is the tax authorities, whose passive involvement in the game precludes the possibility of any action (except for tax reform) which would negate the tax subsidy captured by the coalition.

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[14] Note that the analysis ignores tax consequences at the personal level. Incorporating personal taxation will not affect the main thrust of the argument presented here, though the exact nature of the sub-game between equityholders and convertible bondholders may change. If there are no clientele effects, and the marginal tax rates at the personal level are the same for both classes of securityholders, then the relationship presented here holds if the market value of the tax shield effect is calculated net of personal taxes.

B. Sub-game behavior

Two issues arising out of the dual nature of the convertible, and which may be considered detrimental to existing shareholders' interests, need to be discussed. They are the value of the put option and the relatively high rate of coupon payments when compared to dividends at the time of the primary issue.

As long as the coupon payments provide a higher cash flow to the convertible when compared with an equivalent holding of stock, there is the potential for convertible holders to obtain superior net payoffs when compared with current equityholders. But this superior cash flow does not come to the convertible holders without cost. The quid pro quo is the willingness to pay a conversion premium over existing stock prices for the privilege to convert to equity. Therefore, to some extent the increased current cash flows are being financed with the premium extracted by the firm. It can be argued that the conversion premium is fair compensation for providing the put option to the convertible holder. If this is assumed to be true, there seems to be some basis for believing that the higher coupon payments at the time of the convertible issue serve to transfer wealth from existing equity to the convertible class of security holders (who on conversion will merge with the existing equity category). But there are two factors that ameliorate this seeming inequity.

The coupon payments are fixed and inflexible while dividend payments are not. The firm has the option to raise the dividends at any time in the future. For a firm with a history of periodic increases in dividend payout, this is likely to occur as a natural consequence with the passage of time.[15] The expected *difference* in projected payouts till conversion is the relevant factor. In addition, the tax deductibility of bond coupon payments implies that part of the coupon payment is paid for by the government. Therefore, current equityholders need to be concerned only to the extent that after tax coupon payments are higher than the dividend payments, as the tax subsidy is at the expense of the government treasury. As long as the overall conditions specified earlier are met, both members of the coalition are likely to benefit.

It has been assumed that the put is fairly priced.[16] But this need not be the case. The convertible issue may in fact incorporate a conversion premium which is higher than that permissible in a perfectly competitive market. This is possible because of the absence of a traded put option with the same characteristics as that available to the convertible holder. The key difference is not the extremely large time to maturity of the option when compared with actively traded options, but the precise nature of the put option. The put option's value depends not only on the price of the stock, but on the prevailing term structure of interest rates as well. If interest rates fall, the value of the straight bond feature of the convertible rises, raising the market value floor on the convertible. Conversely, if interest rates rise the market value floor is lowered. Therefore the put value of the option has features which are a result of the hybrid nature of convertibles. The "exercise price" of the put option is not fixed, as is normally the case, but a function of the prevailing interest rates![17]

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[15] In fact, if the dividend is raised substantially, the cash flow implications may be such that convertible bondholders optimal action is voluntary conversion. Therefore a judicious choice of dividend policy can be a viable (though seldom used) method for inducing conversion. Obviously, if such conditions do prevail, equityholders have access to superior expected future cash flows vis-a-vis convertible bondholders.

[16] As mentioned earlier, the convertible could be viewed as a combination of either a straight bond and a call option, or an equity plus put option. The focus is on the put plus equity combination because when the convertible bond is called to induce conversion, it normally trades at a premium to the underlying equity value because of the put feature.

This monopolistic power allows for the possibility of the firm extracting economic rents from the purchasers of the convertible issue. Purchasers may be willing to go along because on balance they need not be worse off than if they had invested in the equity plus equivalent put combination directly! This apparent contradiction is possible in our imperfect world because of two reasons. The first is the extremely large transaction costs that may be involved if market participants tried to create an equivalent stock plus put combination themselves. The second is the distortional impact of corporate taxation, which favours the use of debt financing, all other things being equal. In effect, the tax subsidy available as a result of debt financing makes it possible for the holders of both classes of security, convertible debt and equity, to be better off. The tax subsidy from the use of debt creates additional value for the firm. As long as the contract between the firm and convertible bondholders, both explicit and implicit, is devised in such a way that the sharing of this surplus is perceived to be fair by market participants, there is a strong incentive for both categories of investors to engage in co-operative game playing behavior.

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[17] Note, however, that as the time to maturity of the bond reduces, the exercise price tends to revert to the conversion price of the bond, as on maturity the convertible bond is redeemable at par value.

The sharing rules are not subject to perfect competitive forces. There is thus scope for the judicious exercise of power by the dominant party in the co-operative game. Since the role played by the tax authorities is passive,[18] this implies that market participants may choose to participate in this game by joining the equity category or the convertible category of security holders, on the basis of their individual preferences and subjective assessment of the merits of the two classes of securities. The impact of market participants' actions will be to adjust the relative prices of the two types of investments. Even if the original power seems to rest with the equityholders, who prescribe the rules under which the game is to be played, unrestrained use of that power will have negative long term effects because it will inhibit market participants from entering the game at all! This fact is recognized by the management of firms. The continual need for fresh financing by firms makes this potential threat of future boycott by market participants serious enough to induce the management to play by the implicit rules of the game. Therefore, calls on convertibles are delayed for a long time after it first becomes optimal from equityholders' standpoint to force conversion. In most instances, the convertibles are called only when the put option is so far "out of the money" that the loss of the put protection does not affect the market prices of the convertibles to any great extent.

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[18] Note that holders of other classes of securities, such as straight bonds and preferred shares, are likewise interested observers.

C. Ex post considerations

Though it is not easy to quantify the above argument for implicit contract behavior by firms, it is possible to set forth in general terms the conditions under which existing equityholders would desire forced conversion, and the power of bond markets to enforce the implicit contract. It is assumed that market prices are such that rational investors would convert their bonds to equity if a call notice is received. The relevant time frame for analysis is from the current date to the last date for voluntary conversion, when voluntary conversion is anticipated.

The consequence of a forced conversion will be to transform debt to equity. This implies loss of the coupon deductibility and consequently a drop in the value of the firm to the extent of projected debt tax shields no longer available. Equityholders would therefore find forced conversion beneficial only if the value usurped from the convertible bondholders is greater than this loss in firm value. Convertible bondholders lose the put privilege and the incremental cash flows the coupons provide in excess of the equivalent stock on conversion to equity. Therefore the following condition applies:

Equityholders have an incentive to force conversion if

$$\text{(A)} \qquad \{ TS_{C,t} \} \quad < \quad \{ INT_{C,t} - DIV_{C,t} \} + \{ PUT_{C,t} \}$$

where

subscript "t" implies present values of cash flows are calculated for the time remaining to last date for conversion.

$DIV_{C,t}$ = present value of expected dividend payments to equity shares *due to conversion*, till last conversion date.

Bond markets recognize this incentive for equityholders to renege on the implicit contract. But the bond markets inhibit a forced conversion by threatening to withdraw their goodwill to firms which renege. The loss of goodwill would imply reduced accessibility to the bond markets. This would result in reduced debt tax shield availability in the future. Therefore the bond markets can enforce the implicit contract if:

$$\text{(B)} \quad \{ FTS_t \} + \{ TS_{C,t} \} \geq \{ INT_{C,t} - DIV_{C,t} \} + \{ PUT_{C,t} \}$$

where

FTS_t = present value of expected future tax shields lost due to reduced accessibility to bond markets in future.[19]

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[19] Note that firms may be able to access the bond markets and avail of the debt tax shields, despite the loss of goodwill. However, bond markets will impose more stringent covenants and charge a higher risk premium for future debt issues. Therefore a more general interpretation of this term should be the present value of expected future costs due to non co-operative gaming behavior.

Expression (A) indicates that if the present value of future tax shields available from coupon payments is less than the present value of the net advantage available to bondholders from delaying conversion to the last possible date, there is an incentive to force conversion. However, if condition (B) is met, over the longer term the equityholders have a strong economic incentive to delay calling the convertible bonds. Under these circumstances, it is practical for a firm to force conversion only when the loss of the put protection causes minimal damage to the holders of convertible debt.

If, as argued above, the call feature is not to be used except with the tacit approval of convertible bondholders, what is the need to have it at all? After all, it does not seem to fulfill any economic purpose, but serves to increase the complexity involved in negotiating a successful convertible bond issue. The answer appears to lie not in economic analysis, but in the realm of strategic decision making. Management seems to value the flexibility it obtains in choosing the exact point at which to induce conversion by bondholders. As long as this choice does not significantly diminish the value of the convertibles, the owners of the convertibles appear willing to go along.

V. CONCLUSION

This paper re-examines the "optimal" call policy for convertibles stated by Ingersoll (1977a) and Brennan & Schwartz (1977) from an ex ante standpoint. The role played by market imperfections and taxes are explicitly analyzed. In this setting what was viewed as a two person zero sum game by the authors cited above turns out to be a more complex three person game. The earlier justification for unilateral use of the call provision to force conversion turns out to be overly simplistic. In an extended, repetitive game playing setting, convertible bondholders are not helpless spectators, but active participants in the decision making process. The tax subsidy obtained by using debt instead of equity provides a rationale for the use of convertible bonds as a method of deferred equity financing. The necessity for repeated use of external debt financing by firms, allows the debt markets to enforce co-operative behavior from the management of the firm. As long as equityholders and convertible bondholders can both benefit at the expense of the tax authorities, it is seen that co-operative game playing requires management not to use the call feature in such a manner that the co-operative behavior of market participants is no longer available.

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UTILITY BASED ASSET PRICING

A test of CAPM and beta sensitivity

The Capital Asset Pricing Model (CAPM) continues to be the premier normative relationship employed for the determination of asset returns in a competitive market equilibrium, in both the Sharpe(1964)-Lintner(1965) and Black (1972) versions. The continuing popularity of the CAPM is largely due to the simplicity of the relationship. It is well known that CAPM can be justified by either normality of asset returns or individuals with preferences characterised by quadratic utility. The normality of asset returns is violated by the limited liability of assets, while quadratic utility implies the implausible behavior of increasing risk aversion with increases in wealth. Hence, the theoretical foundations of CAPM are not very strong. The empirical evidence on the validity of CAPM is also rather weak, with the literature not wholly in support of the CAPM position. Representative papers in this area are Fama-MacBeth (1973), Black-Jensen-Scholes (1973), Roll (1977), Litzenberger & Ramaswamy (1979), Gibbons (1982), Jobson & Korkie (1982) and Stambaugh (1982).

The majority of earlier work in this area is based on empirical tests which impose restrictions implied by CAPM on the basic Market Model, which is a very general return generating specification that is consistent with normality of asset returns. In the rest of these studies the CAPM is nested within a multi-factor asset pricing model which would be consistent with either the

Arbitrage Pricing Model due to Ross (1976), or the multi-beta continuous time asset pricing model due to Merton (1973).[1] Thus most of the empirical work has focussed on tests based on models which specify the return generating mechanism.[2]

Preference based models of equilibrium asset pricing, which do not rely on any specific distributional or return generating assumptions, have been extensively discussed in a theoretical setting. Hakansson (1970), Merton (1971, 1973) and Rubinstein (1974) are some of the seminal papers in this area. The Utility Based Asset Pricing Models (UAPM) allow for a richer characterization of the economic environment, even though market clearing constraints restrict the diversity of preferences market participants can exhibit. Despite the substantial body of theoretical knowledge that exists in this area, there have been few attempts to empirically test the UAPM model. Some researchers have tested and compared the UAPM with the CAPM. Representative papers are Roll (1973), Kraus & Litzenberger (1975) and Grauer (1978), who failed to find any difference between CAPM and special cases of the Utility Based Asset Pricing Models. However, because their model specification did not allow for the application of econometric tests, their conclusions were made without any test statistic to back their hypothesis.[3] The earlier work in this area seems to support the view that the UAPM based on isoelastic utility cannot easily be differentiated from the CAPM.

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[1] Merton's (1973) model can be set in the Cox-Ingersoll-Ross (1985) framework. Therefore consistency with this more general model is implicit.

[2] The research cited above is based on stationary return generating models. A new stream of research examines the implications of incorporating time-varying moments to the return distribution. The model allows for the second moments to vary (in a constrained manner), and therefore make it feasible to have conditional expected returns which also vary intertemporally. See Bollerslev, Engle and Woolridge (1988) for a recent application of this type of model.

To this date, there has been a relative absence of statistical tests which seek to distinguish between competing models of asset price equilibrium in a competitive market economy. This paper seeks to fill this void by devising a model structure in which the CAPM (null hypothesis) is nested within a UAPM based on isoelastic power utility (alternative hypothesis). An important feature of the model is that it estimates the Constant Relative Risk Aversion (CRRA) coefficient for the case of Isoelastic Utility endogenously, while also estimating the risk premium demanded of securities as a function of the CRRA of the "composite" individual.[4] In addition, tests will be conducted to ascertain how sensitive the risk parameter (a generalization of the well-known CAPM 'Beta') is to changes in the value of the composite individual's Relative Risk Aversion parameter.

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[3] Most econometric test statistics rely on model specifications which allow the null hypothesis to be formulated as a restriction on the parameter space of the alternate hypothesis. This requires the CAPM to be nested within the more general UAPM framework. Though non-nested tests are theoretically feasible, they are extremely difficult to implement.

[4] See Rubinstein (1974) for a definition.

I UTILITY BASED ASSET PRICING MODEL (UAPM)

The model chosen for test purposes can be obtained from Rubinstein (1974).[5] It arises out of the optimality conditions of the composite individual's intertemporal consumption-investment choice in a time additive expected utility maximization framework. This well known first order Euler condition can be written as

$$E\{ V'(W_1)(R_j - R_F)\} = 0 \qquad (1.1)$$

Eq. (1.1) can be expanded using the definition of covariance to give

$$E(R_j) = R_F + E[V'(W_1)]^{-1} COV\{ r_j, -V'(W_1)\} \qquad (1.1a)$$

where R_j = Stochastic rate of return for the j^{th} asset

R_F = Rate of return on the risk-free asset

r_j = $R_j - R_F$; the excess rate of return for the j^{th} asset

$V(W_1)$ = Derived utility of wealth in the next period

$V'(W_1)$ = Derived marginal utility of wealth in the next period

Note that relationship (1.1a) also applies to the expected rate of return on the market portfolio, R_M, which is just the weighted average of the constituent asset returns. Therefore we have

$$E(R_M) = R_F + E[V'(W_1)]^{-1} COV\{ r_M, -V'(W_1)\} \qquad (1.2)$$

Eq. (1.1a) and eq. (1.2) give on simplification

$$E(R_j) = R_F + b_j[E(R_M) - R_F] \qquad (1.3)$$

where

$$b_j = COV\{ r_j, -V'(W_1)\} / COV\{ r_M, -V'(W_1)\} \qquad (1.3a)$$

Eq. (1.3) is a general utility based equilibrium pricing relationship which is applicable to all economies in which individuals exhibit Linear Risk Tolerance (i.e. have preferences characterized by the HARA class of utility functions), and where the conditions of aggregation are met (Rubinstein,1974). Linear Risk Tolerance implies that risk tolerance, **T**, can be specified as a linear function of wealth, i.e.

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[5] See also Rubinstein(1973) & (1976) for related papers.

$$\mathbf{T} = -V'(W) \,/\, V''(W) = \text{ç} + W \,/\, \mu \tag{1.4}$$

where ç , μ , are preference based parameters.

The derived utility functions which satisfy (1.4) are[6]

$$V(W) = \mu/(1-\mu)\,[\,\text{ç} + W/\mu\,]^{1-\mu} \qquad \textit{for} \quad \mu \neq \infty, 1 \tag{1.4a}$$

$$V(W) = \ln\{\,\text{ç} + W\,\} \qquad \mu = 1 \tag{1.4b}$$

$$V(W) = -\text{ç}\,\mathrm{EXP}\{\,-W/\text{ç}\,\} \qquad \mu = \infty \tag{1.4c}$$

This category of utility functions embodies a diverse spectrum of investor behavior, displaying decreasing ($\mu>0$), increasing ($\mu<0$) and constant ($\mu=\infty$) absolute risk aversion with increase in wealth. Exponential Utility [Eq.(1.4c)], is a limiting case arising when μ goes to infinity and is of no significance for the remainder of the paper. Quadratic Utility, which is a special case of the generalized Power Utility [Eq.(1.4a)] with $\mu = -1$, is of particular importance because it is consistent with the CAPM.

Eq.(1.3) can be simplified further for the case of an economy with a balanced capital account. At the aggregate level, because net borrowing in the economy is zero, the wealth in the next period, W_1, is given by

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[6] Note that the Derived Utility Function is equivalent up to an increasing linear transformation. The parameters, ç, μ, have to be restricted for investors who are risk averse. Specifically, it is required that $\text{ç} \geq 0$ for positive risk aversion at all wealth levels. Additionally, if

(a) $\text{ç} = 0$, then $\mu > 0$.

(b) when $\mu < 0$, for integer values of μ , the function is bounded above by $W = -\mu\text{ç}$. Note that $\text{ç} > 0$ in this case, for wealth to be positive.

$$W_1 = W_0 (1 + R_M) \qquad (1.5)$$

Using Eq.(1.4a) and Eq.(1.5) gives

$$V'(W_1) = [(W_0/\mu)(1 + R_M + Ç)]^{-\mu} \quad \textit{where } Ç = ç\mu/W_0 \qquad (1.6)$$

which along with Eq.(1.3a) gives on simplification[7]

$$b_j = \mathrm{COV}\{ r_j , (1 + R_M + Ç)^{-\mu} \} / \mathrm{COV}\{ r_M , (1 + R_M + Ç)^{-\mu} \} \quad (1.7)$$

It can readily be seen that for Quadratic Utility ($\mu=-1$), Eq.(1.7) reduces to the familiar CAPM 'Beta'. For this special case, the beta relationship is insensitive to changes in the parameter, ç.[8] This allows for considerable simplification of the "generalized" risk parameter given by Eq.(1.7). Because our primary purpose is to test the CAPM within a UAPM framework, we can reduce the complexity of the subsequent empirical work without significantly limiting the results by setting ç=0. This restriction is not without cost, however, as ç=0 implies that Eq.(1.3) is now restricted to the Isoelastic Power Utility Class of derived utility functions (with log utility as a special case when $\mu = 1$). Another consequence of this restriction is that μ has to be greater than zero, for risk averse individuals. The only exception to this is the case of quadratic utility ($\mu = -1$), as the value of the risk parameter in eq.(1.7) is independent of ç, as mentioned earlier. Despite this constraint, the model still allows for a richer range of investor behavior, when compared with the CAPM.[9]

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[7] This relationship was first used by Grauer (1978), who ran single period cross-sectional regressions using exogeneously specified values of ç and μ.

[8] For quadratic utility ($\mu = -1$), the value of the risk parameter, b, is independent of the value of ç. This is because the terms within the covariance operator are linear for $\mu=-1$, and the taste paramater, ç, is a constant.

Risk aversion is imperative for the traditional mean-variance portfolio theory to be meaningful, as the notion of efficient portfolios is based on preferences characterised by risk aversion. UAPM models are not based on any distributional assumptions, and hence asset returns are not restricted to normality. Total risk is no longer characterized by portfolio standard deviation, while the generalized risk parameter given by eq.(1.7) takes on the role of the conventional CAPM beta as a measure of the systematic asset risk which fetches a premium over the riskless rate of return in the market. Ingersoll (1987) discusses the meaning and implication of generalized risk measures in utility based models for equilibrium asset pricing and portfolio separation. The generalized measure of systematic risk, b_j, has some of the desirable properties which the CAPM beta possesses. Portfolio systematic risk is the weighted average of the individual asset's systematic risk. The securities can be ordered uniquely in terms of systematic risk, and this ranking is maintained with respect to all efficient portfolios.[10]

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[9] Utility based models implicitly constrain the expected excess return on the market. This value depends on the aggregated preferences of the market participants. Since the model used in this paper focuses more on the cross-sectional aspects of the risk-return relationship, this intertemporal aspect is not emphasized. For a wider discussion of this issue, see Brown & Gibbons (1985), who estimate μ using time series data on the market returns proxy and the risk free rate of return.

[10] See Ingersoll (1987, p.134) for a formal proof.

Hakansson (1970) has shown that for a constant investment opportunity set, an individual with Isoelastic Utility for consumption will exhibit Isoelastic Derived Utility for wealth. For the more general HARA Utility class, Merton (1971) has shown that HARA Utility for consumption implies HARA Derived Utility, in a continuous time model, where asset prices follow stationary lognormal distributions. In a related paper, Merton (1973) has shown that under these conditions individuals behave as if they were single period optimizers and the CAPM relationship is valid; thereby justifying CAPM under the less onerous lognormal distributional assumption and in a multiperiod setting.

The UAPM model is devoid of any specific distributional assumptions. Merton (1973) has shown that in the case of general return distributions for assets, the portfolio proportions vector of the representative individual is a function of preferences, as indicated by the specific utility function applicable. Thus the market portfolio will be a function of the representative individual's utility. Grauer (1985) has claimed that the key to differentiating between the UAPM and the CAPM relationships lies in examining the weights of the market portfolio, as examination of the risk-return relationship has hitherto been unable to differentiate between the two. Though this line of reasoning is a valid one, it is not possible to put much faith in its fruitfulness, as most empirical work is based on the use of a proxy for the market portfolio. There is no reason to believe that the proxy and the true market portfolio have identical component asset proportions. In addition, a small change in the vector of expected security returns may cause a large shift in the weights of the optimal portfolio, leading to potentially insurmountable measurement error problems. This paper will therefore pursue the more conventional path and attempt to perform a meaningful test of the CAPM within the UAPM framework using the relationship derived earlier.

II. EMPIRICAL MODEL SPECIFICATION

The model developed above is an ex ante model which involves the expectations of rational market participants under conditions in which the basic assumptions of the underlying model are satisfied. The empirical model specification which uses ex post data, allows for a residual error term which captures the difference between ex ante expectations and the ex post realization. In an efficiently functioning market this residual error is expected to exhibit intertemporal independence and have a zero mean, if the expectations are consistent with the underlying economic processes driving the economy. Therefore for the case of Isoelastic Utility, the UAPM becomes in its empirical form :[11]

$$R_{jt} = R_{Ft} + b_j [R_{Mt} - R_{Ft}] + e_{jt} \qquad (2.1)$$

where

$$b_j = COV\{ r_j , (1 + R_M)^{-\mu} \} / COV\{ r_M , (1 + R_M)^{-\mu} \} \qquad (2.1a)$$

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[11] Though eq.(2.1) looks similar in form to the CAPM when tested as a restricted version of the market model, it is important to realize that the econometric implications are very different. The model in this paper restricts b_j explicitly as a function of μ, as specified by eq.(2.1a).

for **securities** $j = 1,2,\ldots\ldots,N$

and **time** $t = 1,2,\ldots\ldots,T$

The residual errors can be characterized as having the following properties (where the NxN contemporaneous covariance matrix of residuals is Σ with typical element given by $\{ s_{jk} \}$)

$$E\{ e_{jt} \} = 0 \qquad \text{for all } j \text{ and } t$$

$$E\{ e_{jv}, e_{kt} \} = s_{jk} \qquad \text{for } v = t\text{ ; } j \text{ and } k \text{ unrestricted}$$

$$= 0 \qquad \text{otherwise.}$$

One final adjustment needs to be made. The model is valid for real returns. Under the assumption that the real risk free rate is constant,[12] it is possible to multiply both the numerator and denominator of Eq. (2.1a) by $(1+R_F)^{-\mu}$ and bring this term inside the covariance operator. With this adjustment the model contains only return differences or ratios, making it suitable for use with either nominal or real returns, as long as the Fisher approximation (i.e. real returns $\approx$ nominal returns - inflation rate) holds.[13] The empirical version of (2.1a) is therefore [14]

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[12] The restrictiveness of this assumption is an empirical issue. For the five year time intervals used in this study, it is reasonable to assume that the burden it imposes is not too objectionable.

[13] Let K = real returns, R = nominal returns, and π = inflation rate.
Then $(1 + R_M) = (1 + K_M)(1 + \pi)$ and $(1 + R_F) = (1 + K_F)(1 + \pi)$.
Dividing the first relationship by the second one we have

$$(1 + R_M)/(1 + R_F) = (1 + K_M)/(1 + K_F).$$

Additionally, if the Fisher approximation holds, i.e. $R = K + \pi$, then $R_j - R_F = K_j - K_F$
This makes the adjusted model independent of the rate of inflation, π, even if inflation is stochastic.

[14] In practice both eq. (2.1a) & (2.1b) were used. The results are not very sensitive to the exact form of the risk factor, b_j, used.

$$b_j = \mathrm{COV}\{\ r_j\ ,\ (1+R_M/1+R_F\ )^{-\mu}\ \}\ /\ \mathrm{COV}\{\ r_M\ ,\ (1+R_M/1+R_F\ )^{-\mu}\ \} \quad (2.1b)$$

As discussed earlier, this model is valid for the class of Isoelastic Utility functions, and reduces to the Sharpe-Lintner one factor CAPM when μ is restricted to -1. The model also permits testing of the Log Utility specification (μ=1) and that implying Risk Neutral asset pricing (μ=0; implying that the risk premium b_j=0 for all "j"). These facts will form the basis of the tests that follow.

Note that the generalized risk parameter, b_j, is a function of the preference parameter, μ. The standard linear regression model is not usable, because μ is to be estimated.[15] No simple linearization of the model is possible without severely limiting the scope of the model, as a truncated Taylor Series expansion will of necessity ignore higher moments.[16] Models such as Kraus & Litzenberger (1976) and Barone-Adesi (1985), which explore the skewness preferences of individuals, are of this type.

Univariate time-series tests using a single risky asset could be carried out in the context of a Nonlinear Regression Model,[17] but this entails a significant loss of available information, thereby reducing the power of the tests. On the other hand single period cross-sectional tests imply use of out of sample data in the calculation of the risk parameters (b_j; $j = 1, \ldots , N$), creating a potentially serious errors-in-variables problem. In an attempt to circumvent these problems the Multivariate Nonlinear Model will be used. The model can be written (in standard vector notation) as

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[15] The risk-return relationship is linear once the generalized risk parameter, b, which is a function of μ, is known. However, because μ is endogeneous to the model, OLS is not a feasible option and nonlinear methods are applied.

[16] Loistl (1976) shows that Taylor series expansion of the expected utility function do not necessarily converge, for the case of power utility. Therefore caution in using such expansions is required.

[17] A number of researchers have used the method of moments technique devised by Hansen & Singleton (1982), to estimate μ using a specific version of the stochastic Euler first order condition. This technique is not applicable to the model formulation specified in this paper. See Ingersoll (1987, pg.135) for a discussion of the use of nonlinear regression to test the model specified in this paper. For a comprehensive coverage of Nonlinear Statistical Models see Gallant (1987).

$$\mathbf{Y} = \mathbf{f}(\ \mathbf{X},\mu\ ) + \mathbf{e} \qquad \ldots \qquad (2.2)$$

with

$$\mathbf{Y} = (\ \mathbf{R}_1'\ \mathbf{R}_2' \quad \mathbf{R}_j' \quad \mathbf{R}_N'\ )'$$

$$\mathbf{X} = (\ \mathbf{X}_1'\ \mathbf{X}_2' \quad \mathbf{X}_j' \quad \mathbf{X}_N'\ )'$$

$$\mathbf{e} = (\ \mathbf{e}_1'\ \mathbf{e}_2' \quad \mathbf{e}_j' \quad \mathbf{e}_N'\ )' \quad \text{.. vectors of size TNx1}$$

where

$$\mathbf{R}_j = (\ R_{j1} \quad R_{j2} \quad R_{jt} \quad R_{jT}\ )'$$

$$\mathbf{X}_j = (\ X_{j1} \quad X_{j2} \quad X_{jt} \quad X_{jT}\ )' \quad \text{.. vectors of size Tx1}$$

and

$$f(\ X_{jt}\ , \mu\ ) = R_{Ft} + b_j\,[\ R_{Mt} - R_{Ft}\ ]$$

also

$$\mathbf{ee}' = \Sigma \otimes \mathbf{I}_T$$

where

Σ = NxN Contemporaneous Covariance Matrix of residuals.

$\mathbf{I}_T$ = Identity Matrix , size TxT

The model as specified above can be conveniently used to obtain the desired test statistic. The null hypothesis implying the CAPM restriction ($\mu = -1$) is nested within the alternative hypothesis, which favours the UAPM. The alternative null hypotheses of Log Utility (μ=1) and Risk Neutrality (μ=0) are also nested within the UAPM, and are interchangeable with the CAPM as the null in the test statistics discussed below. Gallant (1987,chap.5) recommends the use of two statistics, whose merits we discuss below :-

A. *Multivariate Nonlinear Least Squares* (MNLS)

This estimation process does not impose any distributional assumptions on the residuals. Therefore it is more in keeping with the major strength of the UAPM, which imposes no distributional restrictions on the asset returns. The test statistic is :

$$L = \frac{[\, S(\mu_r,\Sigma_u) - S(\mu_u,\Sigma_u)\,] / q}{S(\mu_r,\Sigma_u) / (NT - p)} \qquad (2.3)$$

where $S(\mu,\Sigma) = [\, Y - f(X,\mu)\,]'(\Sigma^{-1} \otimes I_T)[\, Y - f(X,\mu)\,]$

and subscripts r , u represent restricted, unrestricted estimates respectively.

This statistic is asymptotically an F-distribution with "q" degrees of freedom in the numerator and "(NT-p)" degrees of freedom in the denominator. As the null hypothesis in Eq.(2.2) implies 1 restriction on the model, q=1. Because the risk-free rate is part of the independent variables specified in the design matrix, **X**, the parameter space is unidimensional as it contains only μ; therefore p=1. It should be noted that nonlinear least squares methods rely on an asymptotic normal distribution for the residuals. Though this implicitly imposes *conditional* normality on asset return distributions, as is apparent from Eq.(2.2), the *unconditional* asset returns distribution remains unrestricted. As a practical matter, Gallant (1987) recommends that as a precaution, it is prudent to check the residuals for evidence of severe departures from normality.

B . Multivariate Nonlinear Maximum Likelihood (MNML)

This estimation technique makes use of the additional information available from imposing a multivariate normal distribution on the residuals. Most of the applied multivariate tests in the published financial literature incorporate this assumption. Under this model specification, the test statistic becomes :

$$LR = T\{\ln|\Sigma_r| - \ln|\Sigma_u|\} \qquad (2.4)$$

where Σ_r, Σ_u are MLE under H_0(restricted),H_1(unrestricted) respectively.

This statistic is asymptotically $X^2(q)$, and identical to that used by Gibbons (1982). Gallant (1987, p.367) recommends a degree of freedom correction in the form of dividing the right hand side of expression (2.4) by "q" (the number of restrictions imposed by H_0), and suggests evaluating the adjusted statistic using the same F-distribution as in the case of Eq.(2.3) above. Asymptotically the two can be shown to be equivalent. The reason for this preference is that in applied small sample work, the F-distribution seems to work better.

In related work, Jobson & Korkie (1982) have found the use of Bartlett's small sample correction increases the accuracy of their LR statistic. Shanken (1985) has shown that his CSR test, which incorporates an errors-in-variables adjustment and is based on the Hotelling T^2 statistic, bears a simple relationship to the standard LR test. Amsler & Schmidt (1985) use Monte Carlo evidence to confirm the superiority of these statistics over the basic LR test, in their market model setup. These studies highlight the importance of making adjustments for small-sample bias. Monte carlo evidence is desirable to ascertain the form of the correction required. In the absence of direct Monte carlo evidence for the UAPM model specification, this study uses the tests given by Eqs.(2.3) & (2.4), keeping in mind the potential for considerable small sample bias. As Gallant (1987,Chap.5) has shown, asymptotically these two tests are the same. Significant conflict between the two, therefore, is indicative of potential problems in the validity of these tests, as applied to the model under study.

The selection of the appropriate sample size for carrying out the tests is critical in the optimal design of the empirical study. The computational difficulty arising from the nonlinearity of the UAPM model is to some extent offset by the fact that only one parameter needs to be estimated (p=1), and the null hypothesis imposes just one restriction (q=1). Gallant indicates that in some cases as few as 100 d.f. in the denominator are sufficient to impart robustness to the test statistic given by eq.(2.3). The initial choice is to use 20 portfolios and use monthly returns data for a period of 60 months, implying N=20 and T=60. Therefore (NT-p) = 1199, suggesting that the model should be sufficiently robust, given that the denominator degrees of freedom are quite large.[18] However the relationship of N to T may be critical, as shown by Jobson & Korkie (1982). The results which are discussed in the next section do not reveal much cause for concern, however, and therefore backup tests using a smaller number of assets are not carried out.

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[18] For N=20 and T=60, the denominator degrees of freedom are large enough to make the correction suggested by Gallant (1987) unnecessary, as $F(1,1199) \approx X^2(1)$. Therefore Eq. (2.4) and the corresponding $X^2(1)$ distribution, is used in the MNML tests in section IV.

III. METHODOLOGY and RESULTS

The UAPM relationship allows for joint estimation of the Constant Relative Risk Aversion (CRRA) parameter, μ, and the Risk Parameter vector, **b**, of the assets used in the test. Since there is just one parameter to be estimated (namely μ, as **b** is a function of μ), the algorithm used is considerably simplified.[19] IMSL subroutines are used extensively, as standard econometric packages were found unsuitable. Initial univariate exploration using only one risky asset showed that the minimization is not very stable, with a tendency towards taking extreme values of μ. This implies that the minimization surface is not "well-behaved" in the sense that there may be multiple local minimums and non-convexities present. Since there are potential discontinuities in the minimization surface as well (around $\mu=1$ which implies log utility, and $\mu=0$ implying risk neutrality), it is necessary to proceed with caution. As one of the purposes of the paper is to evaluate the sensitivity of the risk parameter to changes in μ, a grid search procedure seems a useful way to proceed.

The tests are carried out with the Ibbotson & Sinquefield data for 1 month T-Bill returns serving as a proxy for the risk-free rate of return, while the NYSE value-weighted Index of monthly returns represents the Market rate of return. The CRSP NYSE monthly returns tape are used for stock returns data. The stocks are grouped into 20 portfolios. To be included a stock has to be listed on the NYSE and have at least 48 returns available in the 60 months prior to the sample period. The portfolios are formed using the Fama & MacBeth (1973) methodology. It is recognized that this grouping procedure will tend to bias the results towards acceptance of the CAPM hypothesis, as the basis of grouping is the out of sample estimate of the risk parameter vector, Beta, that results from the CAPM model. The sample period, T, is set to 60 monthly return observations. This procedure is repeated in 60 month intervals, starting with 1934-1938 as the first sample period. Both test statistics (given by Eq.(2.3) & (2.4)) are evaluated.

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[19] The steps employed in the search procedure are outlined in Annexure A.

A. Tests using portfolio data

The estimates of μ in the univariate estimation using a single portfolio at a time are found to vary widely between different portfolios and are therefore unreliable. They are nevertheless useful in analyzing why the published estimates of CRRA have differed so widely. In order not to detract from the main discussion the univariate results are reported and analyzed separately in Annexure B.

Table 1 shows the results of estimation using the Multivariate Nonlinear Maximum Likelihood (MNML) Estimation technique. The estimates obtained vary from a maximum of 3.506 (for 1949-53) to a minimum of -1.654 (for 1969-73). The restriction implied by CAPM and log utility specifications are accepted at the 95% significance levels (as $X^2(1)$ has a critical value of 3.84) for all sample periods, while the risk neutral specification is not accepted for any sample period. These results seem to be in line with the findings of Roll (1973), Kraus & Litzenberger (1975) and Grauer (1978) as they support the notion that the UAPM cannot be empirically differentiated from the CAPM. For all sample periods, the 95% confidence interval estimates obtained by inverting the X^2 statistic incorporates both $\mu = -1$ and $\mu = 1$ within the acceptance region. It should be noted that the minimization surface exhibits a point discontinuity at $\mu = 0$ (the risk neutral specification), which is therefore excluded from the acceptance region. Fortunately the discontinuity always lies significantly above the minimum point for all sample periods, and can be safely ignored in interpreting the behavior of the test statistic.

Table 2 shows the results obtained using the Multivariate Nonlinear Least Squares (MNLS) Estimation technique. In this procedure two alternate estimates of Σ are used in the weighing scheme. The first involves making a preliminary estimate of μ, assuming i.i.d. residuals across equations.[20] The residuals obtained from this preliminary estimation are used to form an estimate of the residual covariance matrix, Σ, which is then used in the actual estimation. The second uses the OLS residuals from separate univariate regressions to estimate Σ. The reason why the two estimates of Σ differ is that in the first case the value of μ is restricted to be the same across all equations as the multivariate procedure is used, while in the second case no such restriction is imposed, as the estimates of μ are made separately under OLS. The results obtained from the use of the two different estimates of Σ

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[20] This amounts to an unweighted least squares minimization of the residuals.

are similar, with those using OLS residuals conforming more closely to MNML Estimates. Therefore only that set of results is reported.

It can readily be seen that the two procedures give essentially the same results, thereby strengthening the validity of the results and suggesting the sample size may be adequate enough to justify the use of asymptotic test statistics without engaging in Monte Carlo studies. Another positive indication is the fact that unlike the results of the preliminary univariate exploration the minimization surface is well-behaved, with only the point discontinuity at zero interrupting the desirable uniform convexity which is indicative of a unique, global maximum.

B. Tests using individual stock data

In contrast to the results of earlier studies which tested the CAPM against the general Market Model, the CAPM fares extremely well in the more restrictive UAPM framework. The failure to reject the CAPM as the null hypothesis in this paper is even more impressive if one recognizes that simulation evidence suggests that the asymptotic likelihood ratio test statistic is biased towards rejection of the null hypothesis in small samples.[21]

Roll (1977) has pointed out that the use of asset "Betas" for portfolio grouping purposes tends to bias the results towards acceptance of the CAPM hypothesis. In addition, it is well known that normality of asset returns is sufficient for theoretical validity of the CAPM specification.[22] The Central Limit Theorems have shown that the sum of independent non-normally distributed random variates tends to normality asymptotically. Even though asset returns are not completely independent, *most grouping procedures* tend to produce normally distributed portfolio returns.[23] This implicitly biases the estimation results towards the CAPM formulation (and hence $\mu = -1$). To eliminate this potential for biased results, the estimation is replicated using individual securities instead of portfolios.

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[21] See Shanken(1985) for a discussion and Jobson & Korkie (1982), Amsler & Schmidt (1985) for some simulation results.

Since there are a large number of eligible securities for any given sample period, it is not possible to carry out the tests using all the securities simultaneously. The procedure used comprises of first isolating those securities that have no missing returns for the 60 month sample period. These are then ranked according to their CAPM beta estimates using the prior 60 months data (at least 48 returns were required for inclusion). The securities are then split into 20 equally sized groups (with the extra securities being discarded equally at both ends of the ranking). To ensure the maximum spread in the data set and avoid potential complications of near singularities in the residual matrix, the estimation is repeated sequentially using one security (in order of rank) from each group.

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[22] Empirical validity of the CAPM is not assured *even if* returns are normal because of the use of a proxy for the return on the Market.

[23] It should also be noted that monthly returns are used in empirical work largely because they tend to conform most closely with the normality assumption. Therefore the choice of a different period for measuring returns may cause the estimate of CRRA to shift.

This provides a series of estimates for each period, the exact number depending on the eligible securities available.

Table 3 tabulates the results of this procedure. For ease of comparison, the portfolio estimates of μ from table 1 are listed in column 2. Only the MNML Estimates are reported as once again the two estimation techniques give similar results. To avoid needless repetition, in future only MNML estimates will be discussed. The results are reassuring, as the estimate of μ using portfolios is always within two standard deviations of the mean of the μ estimates using individual securities for all sample periods, indicating that a 95% C.I. (assuming an identical, independent, normal distribution of the individual μ estimates) would not reveal any conflict between the two estimates. A normality test on the distribution of estimated μ, for each period, does not reveal any severe departures from the normal distribution.

From the standpoint of investor behavior, risk aversion requires the CRRA parameter, μ, to be positive for an isoelastic utility maximizing individual.[24] In only two periods does the mean of the estimates of μ obtained using individual securities differ from the corresponding portfolio estimate to an extent large enough to reverse signs.

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[24] The sign of μ is important, as risk aversion requires $\mu > 0$ for isoelastic power utility. Note however that for the special case of $\mu = -1$, the model specification is insensitive to the taste parameter ç, and therefore allows for the possibility of ç being > 0 as would be required for risk aversion (see footnote 5). As noted earlier, this is the basis on which $\mu = -1$ makes the model consistent with the CAPM, which requires risk aversion characterized by quadratic utility and hence a value of ç > 0 in addition to $\mu = -1$.

The results reported in Table 3 indicate that grouping procedures do not seem to bias the results in any apparent manner. This is encouraging and allows for further examination of the results without having to worry about a potentially serious threat to the impartiality of the test statistics used in Tables 1 and 2.

In any applied small sample work, the use of asymptotic results is an act of faith. When the model requires the use of non-linear procedures, the potential for biased test results is high. The similarity of results using two different estimation procedures, which are asymptotically identical, reduces the potential for erroneous results. However, there is the possibility of similar biases being present in both estimation techniques. To minimize any remaining doubts, the residuals are inspected. As mentioned earlier, Gallant (1987) suggests that the residuals be examined to ensure that there are no severe departures from normality, even though only asymptotic normality is required.

C. Tests incorporating the zero-beta premium

A close look at the residuals reveal some disturbing features. The residuals show some deviations from normality, which of itself would not be a cause for great concern. However the residual means seem to differ systematically from zero.[25] In most cases (periods when portfolio excess returns tend to be positive), they are positive and of the same order of magnitude as the returns data itself. This is a serious departure from the assumptions of the model, and suggest the possibility of model misspecification. Under the assumption that the sample size is sufficiently large and the CRRA parameter is stationary for the five year sample period used, the residuals should not exhibit any systematic bias, if the model is correctly specified and the market forms expectations rationally. There is the possibility that the bias may be due to the omission of explanatory variables from the model. A two factor model of the type proposed by Black (1972), which incorporates the unknown zero-beta rate of return instead of the risk free rate, may be a more appropriate model.

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[25] The design matrix, X, does not have a column of ones. The absence of the constant term implies that sample residual means cannot be forced to zero, during the minimization of the objective function. What is bothersome is not that the residuals do not have a zero mean, but the observation that on balance they tend to be predominantly positive. This

If this is indeed the case, the situation becomes considerably more complex as there is no clear choice of the appropriate zero-beta rate, R_{zt}, in the context of security returns which are not normally distributed and a market comprised of isoelastic utility maximizers. In fact, even in a mean-variance framework where the market proxy lies within the efficient frontier, there is no uniquely defined R_{zt}.[26]

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can be interpreted as the UAPM version of the well known empirical observation that the intercept, in cross-sectional regressions of the Sharpe-Lintner version of CAPM, is found to be higher than the risk free rate of return.

[26] Usually the minimum variance portfolio orthogonal to the market proxy is chosen for power considerations. This strategy is not a viable one if return distributions are not normal, as total risk is no longer measured by variance (or standard deviation), but by the more general Rothschild-Stiglitz concept of risk. See Ingersoll (1987) for a discussion of this and other related issues.

A constant, deterministic formulation for the zero-beta rate could be estimated endogenously, by simply using another parameter, Z, in the model (eq. 3.1) to replace the nominally risk free rate, R_{Ft}, as was done by Gibbons (1982) in his multivariate market model test of the CAPM. But in a multivariate model in which the sample period extends through 60 months, this formulation is more rigid than is desirable. Since there is no clear theoretical basis for any specific choice of R_{zt}, the simplest alternative is to consider a model which simply adds on a zero-beta premium, Z, to the risk free rate. This choice implies :

$$R_{zt} = R_{Ft} + Z \qquad (3.1)$$

This formulation is desirable from an econometric standpoint as it allows the earlier one factor UAPM model (eq. 2.1) to be nested within this more general two factor UAPM model, thereby allowing tests to ascertain whether the zero-beta premium, Z, is significantly different from zero. The model specification of section II and its empirical counterpart in section III are modified slightly by this introduction of the zero-beta rate, R_z, as there are now two parameters to be estimated endogenously, namely μ and Z . The methodology remains the same except for any degrees of freedom correction to the X^2 test statistic that may be found necessary.

Table 4 lists the estimates of μ and Z, together with the $X^2(1)$ test statistic used to test the null that Z=0. The results show that the test statistic fails to accept the null in only two sample periods. It can be anticipated, therefore, that the introduction of the zero-beta factor in the model formulation does not alter the earlier estimates of μ to any significant extent. Except for the 1954-58 and 1949-53 sample periods, this is indeed the case. For these two periods, the zero-beta premium, Z, is quite large. This tends to reduce the residual means, thereby reducing the potential model misspecification. However, on an overall basis, there is no real gain from the two-factor formulation.

D. Sensitivity of risk parameter to changes in CRRA

The model and the test statistics used have not had much success in separating the competing equilibrium pricing structures under differing CRRA Utility specification. From a practical standpoint, it is desirable to ascertain whether the ranking of various assets, based on their systematic risk, differs significantly as a function of the individual's CRRA parameter, μ.[27] To get an idea of the range for which there is essentially no change in the rankings; the values of μ for which the UAPM asset risk measure, b, has a correlation of 0.95 and 0.99 with corresponding CAPM betas are tabulated (see **TABLE 5**). The data set comprises all eligible individual securities for each sample period. As would be expected, given the foregoing analysis, the results are unspectacular. In all the periods under study, the correlation between the different risk parameters is extremely high for economically plausible values of CRRA. These results tend to confirm the earlier view that at least for Isoelastic Power Utility maximizing individuals, the normative implications for risky stock asset portfolio optimization are not very sensitive to the exact CRRA of the individual, for a broad range of CRRA values.

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[27] The size of the risk premium itself is not very sensitive to the CRRA, μ, as evidenced by the results of the earlier tests.

The estimates of CRRA carried out deserve some comment.[28] The results tend to indicate that the representative Isoelastic Power Utility maximizing individual exhibits both risk averse ($\mu>0$) and risk preferring behavior ($\mu<0$), while risk neutral behavior ($\mu=0$) is strongly rejected. Even when the estimates are close to zero, it should be realized that the point discontinuity at $\mu=0$ discussed earlier indicates that the model does not accept risk neutral behavior. Another aspect that needs to be kept in mind is that the 60 month sample periods may not be sufficiently long for any market cycle effects to be properly worked out of the system, leading to an ex post bias in the parameter estimates. This is an insurmountable problem in a dynamic economy. Though a more extended sample period estimation is econometrically feasible, it is nonetheless inadvisable, given the potential for nonstationarity of the parameters over longer intervals. In keeping with the prior practice in empirical financial research, the 5 year period was chosen arbitrarily. The same methodology using a different time horizon may yield different results.

E. Risk premiums and January seasonality

An interesting application of the methodology used in this paper is to test whether the January seasonality is largely responsible for the risk premiums observed in the marketplace. Tinic & West (1984) have shown that when January returns are excluded, univariate cross-sectional tests of the CAPM do not reject risk-neutrality across stocks. Their findings indicate that there is no statistically significant risk premium for beta, even though the estimate of the intercept term implies a zero beta rate of return significantly higher than the risk free rate as measured by the 1 month T-bill rate as proxy. To test this hypothesis, the MNML estimates of μ are replicated using the one factor model specification of section III, with the January returns excluded. This reduces the number of observations from 60 to 55, and requires degrees of freedom adjustments to the test statistic used. The results are tabulated in **Table 6**. A comparison of these estimates with those in Table 1 shows that the results are essentially unchanged. Risk neutrality is still rejected strongly for all sample periods. Therefore the methodology used here contradicts the earlier study cited above.

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[28] Recent papers which have estimated the CRRA parameter, μ, include Brown & Gibbons (1985), Hansen & Singleton (1982,1983) and Litzenberger & Ronn (1986).

A possible reason for the results in Tinic & West (1984) may be that use of out of sample estimates of the risk premium, and the associated errors-in-variables problem, may have significantly reduced the power of the test to reject risk neutrality. Another source of error may be the inclusion of January returns in obtaining estimates of the required risk premium parameter, beta. If returns exhibit non-stationarity, then it is reasonable to assume that the January risk premium parameters may also be significantly different from the rest of the year. Rogalski & Tinic (1986) show that this is indeed the case, using daily returns on portfolios formed on the basis of firm size. Therefore January return should be excluded in estimating asset risk premiums. The multivariate methodology used here is not sensitive to these problems, since the risk parameter is estimated using in sample data.

IV. CONCLUSION

The Utility Based Asset Pricing Model permits rigorous statistical testing of the Capital Asset Pricing Model to be carried out, in a nonlinear regression framework. Since the model permits only consistent estimates, the results may be subject to unknown small sample bias. The use of two different test statistics produced very similar results, reducing the possibility of substantial undetected bias. There is some scope for concern, however, as the residuals tend to have positive sample means which are of the same order of magnitude as the stock returns themselves. This is indicative of potential model misspecification, and points to the possibility of missing explanatory variables. An effort is made to remedy this problem by relaxing the original single factor model to incorporate a zero-beta premium into the model specification. Only marginal improvement is achieved, as for most sample periods the two-factor model does not differ significantly from the single factor model.

The empirical estimation procedure uses ex post data to estimate the preference based CRRA parameter, μ, which drives the expected risk premiums the market requires of risky assets. In addition, the data set used is such that only the unconditional expectations are incorporated into the model. The parameter estimates will be unbiased only to the extent that the ex ante expectations are on balance realized ex post. Therefore a crucial assumption is that the unconditional expectations and the actual realizations in each period are not systematically different. This is a criterion that is impossible to verify, and therefore all the tests are carried out under the implicit assumption that expectations are rational, and that the model correctly specifies the economic forces endogenous to the model. Earlier papers which conducted tests of the CAPM in a market model setting tended not to accept the CAPM restriction. In the model used here, the CAPM is not rejected for all sample periods. This does not imply that the CAPM is the correct model specification, but merely serves to acknowledge that the Isoelastic UAPM is not a powerful enough alternative to the CAPM, with the methodology and data set used here.

The potential model misspecification is an unresolved issue. This is indicative of the possible existence of missing explanatory variables. The multi-beta CAPM or the APT permit the addition of explanatory variables in an equilibrium setting. However, research has yet to identify and isolate these other economy-wide factors. Until a more powerful model emerges, the tests conducted here indicate that the CAPM continues to be the model of choice as the UAPM fails to dominate the CAPM.

Table 1

MNML Estimation of constant relative risk aversion

Multivariate Nonlinear Maximum Likelihood Estimate of relative risk aversion, μ, for 60 month sample periods. The confidence intervals are obtained by inverting the $X^2(1)$ test statistic.

PERIOD	RELATIVE RISK AVERSION			X^2 (1) TEST STATISTIC		
	Estimate	95% C.I		CAPM	LOG	NEUTRAL
	(μ)	max	min	Ho; $\mu=-1$	$\mu=1$	$\mu=0$
1979-83	-0.180	10.9	-10.9	0.022	0.045	167.9 **
1974-78	-1.019	7.7	-8.3	0.000	0.248	225.0 **
1969-73	-1.654	10.9	-15.7	0.010	0.161	211.3 **
1964-68	-1.633	21.8	-19.6	0.004	0.060	166.8 **
1959-63	0.505	22.6	-13.3	0.033	0.003	190.1 **
1954-58	0.121	21.0	-16.7	0.014	0.008	151.1 **
1949-53	3.506	34.1	-34.9	0.075	0.023	184.1 **
1944-48	1.395	15.4	-14.9	0.102	0.003	250.3 **
1939-43	-0.810	4.2	-6.2	0.005	0.501	219.9 **
1934-38	-0.680	5.5	-7.3	0.009	0.262	196.1 **

** The null hypothesis, Ho, is not accepted at the 95% significance level,as $X^2(1)$ has a critical value of 3.84 .

NOTE : The value of the $X^2(1)$ test statistic is very small, for the CAPM and LOG restrictions. This reflects the fact that the minimization surface is very flat around the optimal value. The parameter, μ, is present both in the numerator and denominator of the risk parameter, b, as a power function. Hence, when the magnitude of μ is small, the model is not very sensitive to changes in the value of μ.

The value of the test statistic is very large for the case of the risk neutral specification because the objective function is discontinuous at $\mu=0$. This implies that the point $\mu=0$ lies outside the acceptance region.

Table 2

MNLS Estimation of constant relative risk aversion

Multivariate Nonlinear Least Squares Estimate of relative risk aversion, μ, for 60 month sample periods. The confidence intervals are obtained by inverting the F test statistic.

PERIOD	RELATIVE RISK AVERSION			F TEST STATISTIC		
	Estimate	95% C.I		CAPM	LOG	NEUTRAL
	(μ)	max	min	Ho: $\mu=-1$	$\mu=1$	$\mu=0$
1979-83	-0.180	11.0	-11.0	0.021	0.042	867.0 * *
1974-78	-1.019	7.9	-8.4	0.000	0.232	2327.1 * *
1969-73	-1.654	11.1	-15.9	0.009	0.150	1842.3 * *
1964-68	-1.636	22.0	-19.7	0.003	0.057	861.2 * *
1959-63	0.505	23.0	-13.5	0.031	0.003	1285.1 * *
1954-58	0.132	21.0	-16.8	0.014	0.008	700.8 * *
1949-53	3.229	33.6	-34.4	0.065	0.018	1216.6 * *
1944-48	1.395	15.4	-15.1	0.096	0.003	3595.1 * *
1939-43	-0.810	4.2	-6.2	0.005	0.471	2140.2 * *
1934-38	-0.681	5.6	-7.4	0.009	0.247	1423.0 * *

* * - The null hypothesis, Ho, is not accepted at the 95% significance level, as the F(1, 1119) statistic has a critical value of 3.84 .

Table 3

MNML estimation of CRRA using individual stocks.

Multivariate Nonlinear Maximum Likelihood estimates using 20 individual securities at a time produce a distribution of estimates for each sample period. The number of estimates, the sample mean & std. dev. is reported. For ease of comparison, the portfolio estimates of CRRA are reproduced from table 1.

PERIOD	RELATIVE RISK AVERSION ESTIMATES (μ				NORMALITY TEST
	PORTFOLIO ESTIMATE	INDIVIDUAL		ESTIMATES	X^2 GOODNESS OF FIT p-value [a]
		number	mean	std. dev	
1979-83	-0.180	53	-0.542	0.599	0.99
1974-78	-1.019	50	-0.988	0.433	0.34
1969-73	-1.654	44	-1.250	0.398	0.80
1964-68	-1.633	39	0.026	1.121	0.91
1959-63	0.505	42	0.018	1.279	0.97
1954-58	0.121	43	0.498	3.569	0.63
1949-53	3.506	39	2.381	4.244	0.69
1944-48	1.395	36	-0.739	1.441	0.87
1939-43	-0.810	30 [b]	-0.885	0.149	0.60
1934-38	-0.680	28 [b]	-0.881	0.462	0.54

[a] The p-value gives the probability of the μ estimates being normally distributed, using the X^2 goodness of fit test statistic.

[b] The number of CRRA estimates is less than that required to ensure that the expected cell size is greater than five, as seven equiprobable cells are used in the X^2 test for normality. This may impart a bias to the test.

Table 4

MNML Estimation incorporating the zero beta premium

The estimates of CRRA, μ, and the zero beta premium, Z, in the two factor model are compared with estimates of CRRA in the one factor model (reproduced from table 1). The one factor model is nested within the two factor model (with Z=0). The $X^2(1)$ test statistic checks for significance of the zero beta premium, Z.

PERIOD	PARAMETER ESTIMATES			TEST STATISTIC
	TWO FACTOR MODEL		1 FACTOR MODEL	
	CRRA μ	Zero-Beta Premium Z (in %)	CRRA μ	Ho: Z = 0
1979-83	-0.523	0.314	-0.180	0.576
1974-78	-1.243	-0.486	-1.019	0.577
1969-73	-1.980	0.378	-1.654	1.346
1964-68	-1.620	0.029	-1.633	0.005
1959-63	-0.756	0.459	0.505	1.248
1954-58	-0.529	1.138	0.121	31.200**
1949-53	-1.863	0.801	3.506	10.801**
1944-48	0.711	0.198	1.395	0.547
1939-43	-1.008	0.583	-0.810	0.851
1934-38	-0.911	0.282	-0.680	0.598

** The restriction that zero-beta premium is zero (i.e. Z=0) is not accepted at the 95% significance level, as $X^2(1)$ has a critical value of 3.84 .

Table 5
Correlation of UAPM risk parameter with CAPM betas

This table lists values of CRRA (μ) which provide a specific level of correlation between the UAPM risk parameter, b, and the CAPM risk parameter, β, for individual securities. Since the CAPM consistent value of CRRA is -1, the listing gives the range[a] of CRRA values around -1 which provide the minimum correlation specified.

	Values of CRRA, μ, which give CORR (b, β) specified				
PERIOD	CORRELATION				DATASET number
	0.95	0.99	0.99	0.95	
1979-83	-12.3	-6.2	4.6	12.2	1360
1974-78	-26.0	-11.0	7.3	16.3	1196
1969-73	-20.1	-9.4	7.8	20.5	996
1964-68	-18.0	-8.6	7.6	21.0	980
1959-63	-24.2	-11.5	9.3	35.4	959
1954-58	-46.0	-19.0	14.6	33.4	958
1949-53	-31.5	-12.5	8.7	20.4	813
1944-48	-13.3	-5.6	2.8	7.1	749
1939-43	-14.8	-7.5	4.6	10.3	645
1934-38	-9.2	-3.9	1.5	4.6	630

[a] Note that for risk neutrality(i.e. $\mu=0$), $b=0$ for all securities and hence CORR(b , β)=0 for this special case. This creates a point discontinuity at $\mu=0$, thereby excluding this value of CRRA from the range specified above.

Table 6

MNML Estimation of CRRA excluding January returns

CRRA (μ) is estimated over 5 year periods using monthly returns except those for the month of January. The procedure is the same as that for results in table 1 with the exception of the exclusion of January returns. This reduces the sample size to 55 observations for each period.

PERIOD	RELATIVE RISK AVERSION			X^2 (1) TEST STATISTIC		
	Estimate (μ)	95% C.I max	min	CAPM Ho: $\mu=-1$	LOG $\mu=1$	NEUTRAL $\mu=0$
1979-83	-0.175	10.3	-10.6	0.247	0.050	162.5 **
1974-78	-0.968	8.3	-7.8	0.000	0.223	203.3 **
1969-73	-1.[illegible]21	11.1	-15.7	0.002	0.120	207.6 **
1964-68	-0.582	21.3	-17.7	0.002	0.023	158.2 **
1959-63	2.189	22.1	-12.1	0.145	0.019	192.7 **
1954-58	-0.336	20.7	-17.4	0.005	0.020	147.3 **
1949-53	0.973	31.4	-39.8	0.013	0.000	180.9 **
1944-48	0.678	15.1	-17.0	0.044	0.002	235.3 **
1939-43	-0.811	4.3	-6.2	0.005	0.479	242.1 **
1934-38	-0.613	5.7	-7.3	0.013	0.228	201.6 **

**The null hypothesis, Ho, is not accepted at the 95% significance level, as $X^2(1)$ has a critical value of 3.84 .

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Annexure A

Grid search algorithm used in MNML estimation routine

1. Choose the size of the initial grid. Let μ take on a value from the grid.

2. Estimate the risk parameter vector, **b**, given the returns data and the assumed value of μ, using Eq. (2.1b).

3. Obtain the residuals vector for each security, using Eq. (2.1).

4. Estimate the matrix of second moments (around zero) for the residuals, Σ, using residual vectors obtained in step 3.

5. Calculate DET{Σ}, the determinant of Σ.

6. Repeat steps 1 to 5 for all values of μ in the grid.

7. Locate the value of μ for which DET{Σ} is the minimum.

8. Iterate steps 1 to 7 using a finer grid around the value of μ for which DET{Σ} was a minimum in the previous iteration.

9. Stop when the change in μ_{min} between iterations is below a predetermined limit.

NOTE:

a) Initial search was confined to $60 \geq \mu \geq -60$ in steps of 1.

b) Convergence limit was set at $\Delta\mu_{min} \leq 0.001$

c) for the MNLS Estimation algorithm, the steps were similar, the differences being necessitated by the change in the objective function.

Annexure B

The univariate estimation of the model specified in section III was carried out for two reasons. The initial OLS estimation was necessary to obtain the residuals and get an estimate of the variance-covariance matrix of residuals, Σ . The second reason involved preliminary searching to get a feel for the type of minimization surface to expect when undertaking the more complex multivariate optimization. As is well known, non-linear optimization is often complicated by the presence of non-convexities and local minima. In addition there is the potential for discontinuities, especially in the kind of model used in this paper, where the parameter μ enters in the numerator and denominator of the risk factor estimate.

These concerns were born out in practice. The OLS optimization was carried out using each of 20 portfolios formed. Multiple minima were observed in a majority of the cases for all sample periods. At times the analysis was further complicated by the fact that the optimization surface tended to be unstable, allowing parameter values to drift to extremes in both the positive and negative direction. The grid search procedure used restricted the search to $60 \geq \mu \geq -60$. In the tables that follow a value of $\mu = \pm 60$ implies that the search was unsuccessful in obtaining a global minimum. In these cases the corner value of μ was used.

The results indicate that the estimate of μ is very sensitive to the data set being used. This is not very surprising as examination of the data showed that the minimization surface had a very small gradient. This allowed the estimate to move around by quite a lot in the presence of outliers in the distribution of asset returns. The parameter μ enters the model as an exponent. Increasing the magnitude of μ tends to increase the skewness of the $(1 + R_{mt})^{-\mu}$ vector. Therefore μ tends to extreme values to compensate for the skewness that outliers induce in the asset returns distribution. Analysis of the data showed that for large values of μ the $(1+R_{mt})^{-\mu}$ vector tends to have just a few values on one side of the mean, indicating a high degree of skewness. For this kind of distribution the outliers in the original market returns vector have a very large weight in the μ estimated. This tends to explain the wide range of estimates of μ produced by studies that have used univariate methodologies. A number of studies have estimated μ to have implausible high values, and the reason seems to be the inherent instability of the univariate estimation process to the presence of outliers.

The multivariate methodology used in this paper reduces the problem discussed above because of the use of 20 portfolio returns to estimate the parameter μ, which is restricted to have the same value in all equations. This reduces the impact of outliers as the multivariate estimation process allocates just a fractional weight to each individual portfolio in the weighted sum squares minimization. Hence the relatively stable estimates obtained.

NOTATION

The univariate results below are reproduced from computer printouts. The notation in the tables should be interpreted as follows :

MYRS=7983 implies	Data set for 1979-83 sample period.
INULL=1	Ho : $\mu = -1$ (i.e. CAPM is valid).
SST	Variance of excess portfolio returns.
RRA1	μ estimate, unrestricted.
RISK1	Risk parameter, b, unrestricted.
SSE1	sum squared errors, unrestricted .

RESTRICTED implies estimates under the CAPM null, i.e. $\mu = -1$.

LR statistic has a critical $X^2(1)$ of 3.84 at the 95% sig. level.

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**************************************************************************************
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                       MSET-637          MYRS-7983          INULL- 1
**************************************************************************************
--------------------------------------------------------------------------------------
                              *****      SUMMARY OF RESULTS      ****
--------------------------------------------------------------------------------------
PORTFOLIO *               *          UNRESTRICTED           *          RESTRICTED            * LR STATISTIC
--------------------------------------------------------------------------------------
NO  BETA  *     SST       *     RRA1       RISK1       SSE1     *    RRA2       RISK2       SSE2     * 60*LN[SSE2/SSE1]
--------------------------------------------------------------------------------------
 1  0.405 *   13.1000     *    2.40913    0.62688     5.59211   * -1.00000    0.61414     5.59548       0.361125E-01
 2  0.589 *   16.3978     *    4.07118    0.81700     3.45800   * -1.00000    0.80326     3.46192       0.678555E-01
 3  0.679 *   15.4285     *    6.37767    0.79987     3.26916   * -1.00000    0.78289     3.27513       0.109513
 4  0.748 *   17.7653     *    1.85695    0.85796     3.45962   * -1.00000    0.84418     3.46355       0.682205E-01
 5  0.806 *   20.4172     *    2.59370    0.93588     3.64457   * -1.00000    0.91502     3.65359       0.140653
 6  0.854 *   17.0378     *    1.94009    0.86846     2.27860   * -1.00000    0.85615     2.28174       0.826810E-01
 7  0.904 *   23.4815     *-57.65803    1.00970     3.84550   * -1.00000    0.99149     3.85237       0.107173
 8  0.956 *   25.5206     *    2.03894    1.07767     2.82731   * -1.00000    1.06203     2.83239       0.107635
 9  1.006 *   28.7670     *    1.48824    1.09156     5.46620   * -1.00000    1.07606     5.47118       0.546690E-01
10  1.060 *   24.8613     *    4.22168    1.03335     3.91516   * -1.00000    1.01968     3.91904       0.593776E-01
11  1.109 *   22.8954     *    2.10819    1.00051     3.58430   * -1.00000    0.98266     3.59091       0.110545
12  1.166 *   26.3561     *    2.49670    1.07022     4.23482   * -1.00000    1.05164     4.24197       0.101273
13  1.228 *   27.8550     *    1.97934    1.07718     5.44986   * -1.00000    1.05840     5.45717       0.804205E-01
14  1.286 *   29.7889     *    2.40539    1.14410     4.44390   * -1.00000    1.12465     4.45174       0.105728
15  1.369 *   26.2123     *    4.49473    1.04175     5.19337   * -1.00000    1.02450     5.19954       0.711893E-01
16  1.446 *   32.3794     *    3.74579    1.11402     8.76322   * -1.00000    1.09086     8.77435       0.761408E-01
17  1.530 *   40.5740     *    2.15477    1.30177     7.67969   * -1.00000    1.28068     7.68890       0.719602E-01
18  1.648 *   37.9214     *    1.72532    1.27335     6.27249   * -1.00000    1.25457     6.27980       0.698849E-01
19  1.821 *   50.7696     *    3.44232    1.41060    12.0323    * -1.00000    1.38900    12.0420        0.481971E-01
20  2.232 *   56.5831     *    2.56300    1.49201    13.4210    * -1.00000    1.46761    13.4334        0.551654E-01
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**************************************************************************************************
**************************************************************************************************
**************************************************************************************************
                     MSET=577          MYRS=7478          INULL= 1
**************************************************************************************************
--------------------------------------------------------------------------------------------------
                              *****      SUMMARY OF RESULTS      ****
--------------------------------------------------------------------------------------------------
PORTFOLIO *                   *          UNRESTRICTED              *          RESTRICTED              * LR STATISTIC
--------------------------------------------------------------------------------------------------
NO  BETA  *    SST            *   RRA1       RISK1      SSE1       *  RRA2      RISK2      SSE2     * 60*LN[SSE2/SSE1]
--------------------------------------------------------------------------------------------------
 1  0.409 *  26.4310   *-42.01682   0.77177    10.7334   * -1.00000   0.77053    10.7334      0.229759E-03
 2  0.628 *  26.8168   * -1.16704   0.87821    6.23970   * -1.00000   0.87749    6.23972      0.130578E-03
 3  0.717 *  22.7383   * -1.21143   0.84905    3.52173   * -1.00000   0.84834    3.52174      0.229049E-03
 4  0.830 *  27.5779   * -1.21282   0.97375    2.25170   * -1.00000   0.97312    2.25171      0.279991E-03
 5  0.900 *  26.1237   * -1.21744   0.94030    2.67300   * -1.00000   0.93920    2.67303      0.725896E-03
 6  0.973 *  28.1093   *-20.15401   0.97555    2.83851   * -1.00000   0.97459    2.83854      0.556635E-03
 7  1.040 *  30.1525   * -3.53597   0.95634    5.94433   * -1.00000   0.95563    5.94436      0.290406E-03
 8  1.109 *  33.1546   * -1.41721   1.03043    5.45280   * -1.00000   1.02852    5.45290      0.107643E-02
 9  1.171 *  36.7216   *  1.87559   1.03439    8.93161   * -1.00000   1.03217    8.93174      0.885133E-03
10  1.231 *  42.6855   *  0.08542   1.13984    8.51448   * -1.00000   1.13794    8.51455      0.884785E-03
11  1.293 *  47.7389   *  0.14816   1.15577    12.8375   * -1.00000   1.15356    12.8376      0.612085E-03
12  1.345 *  43.0029   *  1.54573   1.14013    9.28871   * -1.00000   1.13976    9.28875      0.304166E-03
13  1.413 *  41.9058   *  3.55194   1.05746    13.3784   * -1.00000   1.05645    13.3785      0.573285E-03
14  1.492 *  53.7759   * -1.33098   1.23154    14.6849   * -1.00000   1.23154    14.6849      0.452753E-05
15  1.573 *  54.4923   *  2.82834   1.26256    13.6250   * -1.00000   1.26098    13.6252      0.913780E-03
16  1.657 *  57.9392   * 20.77961   1.25542    17.8818   * -1.00000   1.25200    17.8819      0.105536E-02
17  1.751 *  65.2370   * -2.10513   1.23522    25.9562   * -1.00000   1.23223    25.9564      0.555826E-03
18  1.875 *  71.1920   * -0.21040   1.30399    28.8562   * -1.00000   1.29975    28.8566      0.100123E-02
19  2.040 *  68.5690   *  4.10858   1.28603    27.0073   * -1.00000   1.28325    27.0077      0.847097E-03
20  2.367 *  102.960   * 11.75606   1.56134    41.3197   * -1.00000   1.55677    41.3203      0.813555E-03
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                          MSET-517            MYRS-6973            INULL= 1
*******************************************************************************************************
-------------------------------------------------------------------------------------------------------
                                  *****       SUMMARY OF RESULTS       ****
-------------------------------------------------------------------------------------------------------
PORTFOLIO *                 *          UNRESTRICTED                 *          RESTRICTED                 * LR STATISTIC
-------------------------------------------------------------------------------------------------------
NO  BETA  *    SST          *    RRA1      RISK1        SSE1        *    RRA2      RISK2        SSE2      * 60*LN[SSE2/SSE1]
-------------------------------------------------------------------------------------------------------
 1  0.276 *  11.7898        *  0.08922    0.65762      3.53591      * -1.00000   0.65794      3.53591       0.323075E-04
 2  0.488 *  16.5903        * -2.01822    0.87847      1.95181      * -1.00000   0.87658      1.95188       0.209018E-02
 3  0.607 *  16.2835        * -3.77614    0.83207      3.13996      * -1.00000   0.83052      3.14000       0.884342E-03
 4  0.710 *  22.3729        * -2.66221    0.99689      3.49348      * -1.00000   0.99521      3.49353       0.926985E-03
 5  0.779 *  19.6069        * 24.93187    0.97202      1.83518      * -1.00000   0.96753      1.83557       0.126337E-01
 6  0.852 *  23.7529        * -6.60616    1.05901      2.44484      * -1.00000   1.05733      2.44489       0.132660E-02
 7  0.931 *  29.6107        * -3.39403    1.20733      2.27042      * -1.00000   1.20095      2.27120       0.206508E-01
 8  0.987 *  33.3547        * -2.62553    1.25769      3.70242      * -1.00000   1.25088      3.70331       0.144119E-01
 9  1.046 *  28.7350        * -2.91644    1.19132      1.76910      * -1.00000   1.18949      1.76916       0.217428E-02
10  1.110 *  32.6065        * -3.84360    1.21693      4.31964      * -1.00000   1.21810      4.31967       0.367419E-03
11  1.187 *  37.2914        *-47.35040    1.32406      4.04313      * -1.00000   1.32112      4.04330       0.245249E-02
12  1.247 *  32.9081        *-55.17371    1.23200      4.04564      * -1.00000   1.23050      4.04568       0.642231E-03
13  1.311 *  36.3769        *-27.68797    1.29444      5.14167      * -1.00000   1.26581      5.14299       0.166445E-01
14  1.371 *  35.0213        * -3.02196    1.26465      5.02686      * -1.00000   1.25768      5.02774       0.104721E-01
15  1.439 *  38.8022        * -2.20471    1.33775      4.93452      * -1.00000   1.33367      4.93481       0.350421E-02
16  1.531 *  49.5208        * -2.09370    1.48644      7.72273      * -1.00000   1.48195      7.72312       0.300222E-02
17  1.618 *  46.3612        * -5.23919    1.48353      5.62726      * -1.00000   1.47158      5.63002       0.291475E-01
18  1.738 *  60.5315        *-54.25801    1.65103      9.23873      * -1.00000   1.64354      9.23980       0.698363E-02
19  1.903 *  61.5388        * -4.98162    1.67430      9.10584      * -1.00000   1.66425      9.10778       0.127401E-01
20  2.287 *  77.0154        * -5.92413    1.83861     14.3669       * -1.00000   1.82422     14.3708        0.165738E-01
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**************************************************************************************************
                        MSET-467          MYRS-6468          INULL= 1
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--------------------------------------------------------------------------------------------------
                              *****      SUMMARY OF RESULTS      ****
--------------------------------------------------------------------------------------------------
PORTFOLIO *                 *          UNRESTRICTED              *            RESTRICTED             * LR STATISTIC
--------------------------------------------------------------------------------------------------
NO  BETA  *    SST          *   RRA1        RISK1      SSE1      *   RRA2       RISK2      SSE2      * 60*LN[SSE2/SSE1]
--------------------------------------------------------------------------------------------------
  1 0.325 *  9.66853        *  20.69873    0.81691    5.80648    *  -1.00000   0.74433    5.85711      0.520862
  2 0.538 *  8.49637        *  29.96448    0.87037    2.41571    *  -1.00000   0.83308    2.42907      0.330976
  3 0.618 *  9.05028        *  17.31359    0.89138    2.48306    *  -1.00000   0.85900    2.49314      0.242910
  4 0.669 *  10.4284        *  13.50077    0.95293    2.75272    *  -1.00000   0.92359    2.76100      0.180037
  5 0.719 *  12.3339        *-22.97938    1.03328    3.58895    *  -1.00000   0.99355    3.60412      0.253069
  6 0.774 *  9.77913        *-30.76763    0.97080    1.94063    *  -1.00000   0.93696    1.95163      0.339143
  7 0.820 *  11.8082        *-24.73244    1.04166    2.66517    *  -1.00000   1.00873    2.67559      0.234160
  8 0.864 *  11.3717        *-21.40572    1.05706    1.76592    *  -1.00000   1.02937    1.77328      0.249842
  9 0.910 *  12.7768        *  38.78698    1.07183    3.05180    *  -1.00000   1.03918    3.06205      0.201075
 10 0.952 *  16.1227        *  42.86476    1.27738    3.11924    *  -1.00000   1.22073    3.15008      0.590262
 11 1.002 *  16.3370        *-27.26668    1.27799    2.70278    *  -1.00000   1.23463    2.72086      0.399890
 12 1.052 *  15.0993        *  53.74700    1.20805    3.38127    *  -1.00000   1.15638    3.40693      0.453559
 13 1.099 *  14.8306        *-25.81466    1.18740    2.34515    *  -1.00000   1.18415    2.34777      0.669143E-01
 14 1.163 *  15.0639        *-39.70651    1.29438    3.63183    *  -1.00000   1.26120    3.64241      0.174541
 15 1.239 *  19.7074        *-55.78890    1.30172    5.05079    *  -1.00000   1.28180    5.06476      0.165721
 16 1.318 *  19.5034        *-19.94190    1.35472    4.14660    *  -1.00000   1.30955    4.16621      0.283040
 17 1.404 *  26.0415        *-13.01479    1.54569    6.13914    *  -1.00000   1.49242    6.16641      0.265941
 18 1.510 *  29.6205        *-60.00000    1.63401    6.63814    *  -1.00000   1.59212    6.65563      0.157861
 19 1.640 *  24.4885        *  34.50945    1.47948    6.68369    *  -1.00000   1.42041    6.71723      0.300320
 20 1.936 *  36.4419        *  29.65865    1.59292    15.0406    *  -1.00000   1.55184    15.0713      0.122061
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                    | MSET=397           MYRS=5963            INULL= 1
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------------------------------------------------------------------------------------------------
                        *****      SUMMARY OF RESULTS      ****
------------------------------------------------------------------------------------------------
PORTFOLIO *                *            UNRESTRICTED              *            RESTRICTED              *  LR STATISTIC
------------------------------------------------------------------------------------------------
NO  BETA  *    SST       *    RRA1       RISK1        SSE1      *    RRA2       RISK2        SSE2      *  60*LN(SSE2/SSE1)
------------------------------------------------------------------------------------------------
 1  0.164 *  10.1077     * 35.64595    0.76379      2.36059     *  -1.00000   0.75084      2.36299       0.608230E-01
 2  0.332 *  10.9430     * 25.42299    0.77329      3.01103     *  -1.00000   0.75995      3.01358       0.506608E-01
 3  0.414 *  11.5031     *-12.04611    0.81450      2.53181     *  -1.00000   0.80542      2.53298       0.279068E-01
 4  0.499 *  13.4709     * 12.19930    0.89618      2.63957     *  -1.00000   0.88535      2.64124       0.380937E-01
 5  0.556 *  13.0771     *-21.02271    0.86459      2.94456     *  -1.00000   0.85566      2.94570       0.232305E-01
 6  0.611 *  16.4722     * 50.71456    0.98435      3.12176     *  -1.00000   0.98046      3.12197       0.414080E-02
 7  0.677 *  15.8295     * -5.90666    1.00194      2.04328     *  -1.00000   0.99657      2.04369       0.120985E-01
 8  0.742 *  13.4039     *  0.99345    0.90050      2.06294     *  -1.00000   0.90358      2.06307       0.392869E-02
 9  0.811 *  15.4558     * 47.24741    0.93094      3.38850     *  -1.00000   0.93192      3.38851       0.244717E-03
10  0.880 *  19.8293     *  2.53587    1.07987      3.37314     *  -1.00000   1.08978      3.37454       0.249723E-01
11  0.944 *  14.8816     * -4.01461    0.93446      2.87040     *  -1.00000   0.93008      2.87068       0.571700E-02
12  1.002 *  17.7566     * -3.24773    1.07807      1.75238     *  -1.00000   1.07351      1.75268       0.101665E-01
13  1.062 *  16.5824     *  2.13282    1.01239      2.15856     *  -1.00000   1.01979      2.15935       0.217627E-01
14  1.121 *  18.7391     * -0.05169    1.09005      2.17028     *  -1.00000   1.09197      2.17033       0.144770E-02
15  1.175 *  17.2255     *  0.59930    1.01788      2.66913     *  -1.00000   1.02419      2.66970       0.127908E-01
16  1.238 *  21.2455     *  0.39709    1.14578      2.85383     *  -1.00000   1.15084      2.85420       0.766983E-02
17  1.317 *  24.5624     * -0.28772    1.22704      3.52123     *  -1.00000   1.23071      3.52142       0.327713E-02
18  1.404 *  21.0482     *  1.07046    1.08237      4.56116     *  -1.00000   1.09028      4.56205       0.117487E-01
19  1.542 *  26.5086     *  2.86108    1.25311      4.24031     *  -1.00000   1.27004      4.24440       0.578770E-01
20  1.835 *  29.5950     *  3.57479    1.23814      7.72233     *  -1.00000   1.26580      7.73327       0.848725E-01
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****************************************************************************************************
                     MSET=337          MYRS=5488          INULL= 1
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----------------------------------------------------------------------------------------------------
                              *****     SUMMARY OF RESULTS     ****
----------------------------------------------------------------------------------------------------
PORTFOLIO *                  *           UNRESTRICTED                  *          RESTRICTED               * LR STATISTIC
----------------------------------------------------------------------------------------------------
NO  BETA  *    SST           *    RRA1        RISK1         SSE1       *   RRA2       RISK2      SSE2      * 60*LN[SSE2/SSE1]
----------------------------------------------------------------------------------------------------
 1  0.280 *  3.74339         *  53.34045    0.42009       2.08088      * -1.00000   0.41029    2.09756       0.478973
 2  0.440 *  4.62263         *-60.00000     0.49077       2.31127      * -1.00000   0.46869    2.34631       0.902822
 3  0.553 *  7.32644         *  17.65851    0.62262       2.96692      * -1.00000   0.61616    2.97564       0.176266
 4  0.635 *  7.99310         *-60.00000     0.68641       2.47347      * -1.00000   0.68407    2.47779       0.104817
 5  0.716 *  8.21396         *  35.31280    0.70879       2.74620      * -1.00000   0.67813    2.75973       0.294883
 6  0.796 *  9.27758         *-46.10098     0.80586       2.45800      * -1.00000   0.76081    2.48720       0.708686
 7  0.878 *  10.4033         *-37.65007     0.80231       3.55702      * -1.00000   0.76102    3.58157       0.412644
 8  0.954 *  11.7029         *  48.26521    0.89498       2.66529      * -1.00000   0.86875    2.67519       0.222455
 9  1.018 *  11.5085         *  58.53415    0.90887       2.22046      * -1.00000   0.88100    2.23166       0.301392
10  1.071 *  14.7937         *  41.71670    0.98648       3.26376      * -1.00000   0.97946    3.26793       0.766157E-01
11  1.126 *  14.1303         *-60.00000     0.99102       2.83330      * -1.00000   0.97087    2.84295       0.204092
12  1.181 *  14.1947         *  60.00000    1.03280       2.51474      * -1.00000   0.99068    2.54189       0.644241
13  1.228 *  17.4755         *   3.83331    1.04263       4.74484      * -1.00000   1.02876    4.74761       0.350203E-01
14  1.287 *  18.4092         *  -0.90734    1.04714       5.20894      * -1.00000   1.04699    5.20894       0.364769E-05
15  1.359 *  17.4758         *   1.88575    1.07435       3.65773      * -1.00000   1.07127    3.65787       0.223387E-02
16  1.444 *  20.4670         *  15.52423    1.14771       4.21580      * -1.00000   1.16232    4.21887       0.437144E-01
17  1.553 *  25.9733         *-10.36770     1.29294       6.05312      * -1.00000   1.28630    6.05376       0.630195E-02
18  1.660 *  22.8050         *   0.98852    1.22109       4.82456      * -1.00000   1.22205    4.82457       0.166020E-03
19  1.802 *  22.9073         *  -0.33679    1.20010       5.60744      * -1.00000   1.19860    5.60747       0.346600E-03
20  2.111 *  35.1647         *-16.65575     1.39432       10.6910      * -1.00000   1.42808    10.7074       0.919681E-01
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****************************************************************************************************
                          MSET=277          MYRS=4953          INULL= 1
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----------------------------------------------------------------------------------------------------
                                *****     SUMMARY OF RESULTS      ****
----------------------------------------------------------------------------------------------------
PORTFOLIO *              *         UNRESTRICTED                 *           RESTRICTED            * LR STATISTIC
----------------------------------------------------------------------------------------------------
NO  BETA  *    SST       *    RRA1       RISK1        SSE1      *   RRA2       RISK2       SSE2   *  60*LN(SSE2/SSE1)
----------------------------------------------------------------------------------------------------
 1  0.454 *  4.48508     *   7.19244   0.56809     1.60782     * -1.00000   0.56025    1.60847     0.244095E-01
 2  0.660 *  4.42061     *  14.04066   0.63049     0.973559    * -1.00000   0.61435    0.976310    0.170546
 3  0.774 *  6.88921     *  18.69811   0.76670     1.70223     * -1.00000   0.75264    1.70433     0.739264E-01
 4  0.844 *  6.75661     *  52.32117   0.79742     0.986396    * -1.00000   0.79290    0.986613    0.132340E-01
 5  0.911 *  7.46603     *  16.85134   0.83152     1.36606     * -1.00000   0.81622    1.36857     0.109198
 6  0.982 *  11.1786     *  -7.02447   0.98227     2.26976     * -1.00000   0.98514    2.26985     0.230186E-02
 7  1.055 *  11.4040     *-20.54018    1.05763     1.08918     * -1.00000   1.06057    1.08984     0.354113E-01
 8  1.121 *  12.7154     *  -2.89103   1.07825     1.97405     * -1.00000   1.08173    1.97418     0.390885E-02
 9  1.165 *  12.2974     *   0.36029   1.08393     1.52500     * -1.00000   1.08326    1.52501     0.183525E-03
10  1.220 *  12.6310     *-29.87295    1.07658     1.85285     * -1.00000   1.08466    1.85524     0.774668E-01
11  1.279 *  14.1591     *-16.13606    1.11998     2.41314     * -1.00000   1.13159    2.41458     0.358181E-01
12  1.339 *  14.7842     *-41.97470    1.16987     1.92724     * -1.00000   1.18404    1.92937     0.663946E-01
13  1.401 *  18.2592     *  59.43688   1.26180     3.17661     * -1.00000   1.28308    3.18142     0.908377E-01
14  1.459 *  18.7950     *-60.00000    1.27412     3.67764     * -1.00000   1.28901    3.68904     0.185592
15  1.512 *  19.5547     *-60.00000    1.31920     3.41362     * -1.00000   1.33851    3.43632     0.397681
16  1.572 *  18.8618     *   2.93204   1.32250     2.92284     * -1.00000   1.32275    2.92305     0.428707E-02
17  1.651 *  22.5571     *  60.00000   1.36640     5.15975     * -1.00000   1.38110    5.16988     0.117597
18  1.744 *  25.6602     *-60.00000    1.37950     6.77498     * -1.00000   1.44985    6.83927     0.566690
19  1.890 *  28.0627     *-60.00000    1.52170     6.22813     * -1.00000   1.56056    6.28318     0.528006
20  2.214 *  32.8362     *-60.00000    1.55609     9.59612     * -1.00000   1.61510    9.68588     0.558586
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************************************************************************************************************
************************************************************************************************************
************************************************************************************************************
                        MSET=217          MYRS=4448          INULL= 1
************************************************************************************************************
------------------------------------------------------------------------------------------------------------
                              *****    SUMMARY OF RESULTS    ****
------------------------------------------------------------------------------------------------------------
                                                            *          RESTRICTED           *  LR STATISTIC
PORTFOLIO *                 *      UNRESTRICTED
------------------------------------------------------------------------------------------------------------
NO  BETA  *   SST        *    RRA1       RISK1      SSE1     *   RRA2       RISK2      SSE2   *  60*LN(SSE2/SSE1)
------------------------------------------------------------------------------------------------------------
 1  0.500 *   8.66321    *  10.00740    0.68615    1.74348   * -1.00000    0.67664    1.74510     0.555687E-01
 2  0.642 *  15.5189     *  12.37511    0.89374    2.20536   * -1.00000    0.87837    2.20967     0.116902
 3  0.731 *  17.5677     *  -5.83395    0.92856    2.68126   * -1.00000    0.92774    2.68163     0.832851E-02
 4  0.813 *  17.4474     *  52.89819    0.97115    1.46074   * -1.00000    0.96055    1.46278     0.839525E-01
 5  0.897 *  16.7540     *  10.28765    0.96133    1.19823   * -1.00000    0.94823    1.20136     0.156327
 6  0.978 *  21.5714     *  -5.89001    1.08922    1.11494   * -1.00000    1.08523    1.11523     0.156367E-01
 7  1.066 *  22.7737     * -43.68716    1.12359    1.18757   * -1.00000    1.11530    1.18881     0.630403E-01
 8  1.141 *  23.8581     *   7.43232    1.13876    1.97159   * -1.00000    1.12588    1.97336     0.539756E-01
 9  1.218 *  26.8138     *   1.21649    1.25081    1.83680   * -1.00000    1.24625    1.83718     0.123771E-01
10  1.301 *  30.8296     * -43.05106    1.28184    2.97771   * -1.00000    1.26783    2.98128     0.719830E-01
11  1.381 *  29.4204     *   0.41456    1.23688    2.86548   * -1.00000    1.23685    2.86549     0.182635E-03
12  1.484 *  33.9394     * -22.70041    1.34691    2.63768   * -1.00000    1.34241    2.63805     0.839128E-02
13  1.604 *  35.0016     *  24.27417    1.37060    2.50514   * -1.00000    1.36767    2.50530     0.376428E-02
14  1.699 *  36.7638     *   0.88428    1.37404    4.24335   * -1.00000    1.36838    4.24393     0.824152E-02
15  1.806 *  42.7646     * -20.29694    1.44677    6.44187   * -1.00000    1.44589    6.44188     0.129412E-03
16  1.924 *  43.7477     *  23.71393    1.50576    4.23979   * -1.00000    1.50797    4.23988     0.126083E-02
17  2.035 *  45.0014     *   6.26245    1.55772    3.51140   * -1.00000    1.54626    3.51379     0.408059E-01
18  2.223 *  60.1290     *  -0.80447    1.74361    7.31621   * -1.00000    1.74347    7.31621     0.174467E-04
19  2.816 *  79.6799     *   3.94296    1.88513   18.2923    * -1.00000    1.87987   18.2928      0.165387E-02
20  3.243 *  70.2885     *   1.66342    1.68355   21.8632    * -1.00000    1.67061   21.8662      0.835767E-02
------------------------------------------------------------------------------------------------------------
```

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*********************************************************************************************
*********************************************************************************************
*********************************************************************************************
                    MSET=157        MYRS=3943        INULL= 1
*********************************************************************************************
---------------------------------------------------------------------------------------------
                         *****     SUMMARY OF RESULTS     ****
---------------------------------------------------------------------------------------------
PORTFOLIO *                 *          UNRESTRICTED            *          RESTRICTED            * LR STATISTIC
---------------------------------------------------------------------------------------------
NO  BETA  *    SST          *   RRA1      RISK1       SSE1     *   RRA2     RISK2      SSE2     * 60*LN[SSE2/SSE1]
---------------------------------------------------------------------------------------------
 1  0.348 *  17.3809        * -0.39784   0.61724     5.81482   * -1.00000  0.61095    5.81607     0.129029E-01
 2  0.598 *  20.6079        * -0.24661   0.74884     3.43244   * -1.00000  0.74288    3.43356     0.196085E-01
 3  0.724 *  23.8030        *  0.11397   0.82252     3.55998   * -1.00000  0.81155    3.56378     0.640467E-01
 4  0.848 *  28.1283        *  1.23742   0.91868     2.47682   * -1.00000  0.90958    2.47944     0.635318E-01
 5  0.970 *  37.3502        *  0.66978   1.03911     4.85730   * -1.00000  1.02789    4.86129     0.492153E-01
 6  1.083 *  49.6022        *  7.51084   1.20365     5.03503   * -1.00000  1.19552    5.03713     0.249330E-01
 7  1.180 *  53.5978        * -3.55401   1.25052     5.69734   * -1.00000  1.24059    5.70046     0.328939E-01
 8  1.266 *  76.9684        * -2.15281   1.44177     13.5348   * -1.00000  1.42888    13.5397     0.232904E-01
 9  1.353 *  56.3454        *-24.84628   1.27027     7.08792   * -1.00000  1.25904    7.09190     0.337145E-01
10  1.432 *  83.1220        * -2.01578   1.46410     17.4775   * -1.00000  1.45237    17.4818     0.149487E-01
11  1.513 *  89.5587        * -1.79776   1.55650     15.4635   * -1.00000  1.54349    15.4689     0.207944E-01
12  1.601 *  95.0569        * -1.29004   1.61731     13.7101   * -1.00000  1.61213    13.7110     0.371760E-02
13  1.680 *  124.280        * -1.58236   1.81690     23.1993   * -1.00000  1.80227    23.2061     0.175107E-01
14  1.765 *  153.144        * -1.56093   1.89455     44.4711   * -1.00000  1.87403    44.4844     0.179812E-01
15  1.844 *  139.837        * -1.40799   1.87583     31.0229   * -1.00000  1.86615    31.0258     0.572946E-02
16  1.918 *  165.989        * -1.39203   1.99318     44.1142   * -1.00000  1.97814    44.1214     0.973737E-02
17  1.986 *  255.317        * -1.34158   2.30983     91.2754   * -1.00000  2.29385    91.2835     0.531430E-02
18  2.062 *  258.466        * -1.37940   2.40441     81.2211   * -1.00000  2.38585    81.2320     0.804751E-02
19  2.188 *  213.801        * -1.38962   2.08828     80.6802   * -1.00000  2.06974    80.6910     0.808259E-02
20  2.553 *  284.009        * -1.14247   2.43901     79.0163   * -1.00000  2.43110    79.0183     0.150262E-02
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*************************************************************************************************************
*************************************************************************************************************
*************************************************************************************************************
                        NSET= 97          MYRS=3438          INULL= 1
*************************************************************************************************************
-------------------------------------------------------------------------------------------------------------
                             *****     SUMMARY OF RESULTS      ****
-------------------------------------------------------------------------------------------------------------
PORTFOLIO *                *           UNRESTRICTED                *            RESTRICTED                 *  LR STATISTIC
-------------------------------------------------------------------------------------------------------------
NO  BETA  *    SST         * .  RRA1       RISK1        SSE1       *   RRA2       RISK2        SSE2        *  60*LN[SSE2/SSE1]
-------------------------------------------------------------------------------------------------------------
  1  0.361 *   26.8647     *   4.03096    0.66154      6.57311     * -1.00000    0.64702      6.58381         0.975723E-01
  2  0.630 *   45.4068     *   0.01079    0.87853      7.32762     * -1.00000    0.87698      7.32774         0.987497E-03
  3  0.743 *   53.3495    .* -8.32789    0.96077      6.60233     * -1.00000    0.97301      6.60562         0.298835E-01
  4  0.827 *   58.7646     * -1.67807    1.02859      6.38906     * -1.00000    1.02840      6.38912         0.304515E-03
  5  0.911 *   67.3718     * -1.05299    1.11295      6.05936     * -1.00000    1.11274      6.05937         0.223419E-04
  6  0.980 *   65.0165     *   7.13157    1.08159      8.14804     * -1.00000    1.07404      8.15362         0.410612E-01
  7  1.045 *   88.5912     * -7.59297    1.25955      11.3561     * -1.00000    1.26061      11.3613         0.272851E-01
  8  1.129 *   115.679     *-38.77135    1.43776      14.3332     * -1.00000    1.43123      14.3354         0.906674E-02
  9  1.216 *   120.413     *-10.25265    1.47365      12.4375     * -1.00000    1.47679      12.4380         0.240571E-02
 10  1.289 *   116.118     * -0.98457    1.43523      14.1092     * -1.00000    1.43527      14.1092         0.408963E-06
 11  1.373 *   133.885     * -5.13029    1.60099      8.11863     * -1.00000    1.59426      8.12092         0.169400E-01
 12  1.461 *   172.469     *-14.16482    1.68105      33.7542     * -1.00000    1.67425      33.7566         0.421504E-02
 13  1.545 *   133.279     * -7.96555    1.57006      11.4551     * -1.00000    1.56852      11.4553         0.644626E-03
 14  1.616 *   163.875     *-12.75405    1.68575      22.7747     * -1.00000    1.68872      22.7751         0.117715E-02
 15  1.697 *   155.909     * -0.89696    1.64307      22.1192     * -1.00000    1.64371      22.1193         0.567809E-04
 16  1.779 *   195.894     * -1.70320    1.88063      22.5977     * -1.00000    1.87163      22.6018         0.109145E-01
 17  1.888 *   213.433     *-12.09678    1.91598      30.3804     * -1.00000    1.92324      30.3831         0.527593E-02
 18  1.983 *   246.718     *-12.12137    2.09054      32.0540     * -1.00000    2.08269      32.0571         0.584941E-02
 19  2.147 *   207.920     * -1.00790    1.90422      28.3792     * -1.00000    1.90413      28.3792         0.871463E-06
 20  2.656 *   264.610     *-12.98650    2.03923      61.2856     * -1.00000    2.02766      61.2924         0.664506E-02
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Relative Risk Aversion Estimation

The pioneering work of Hakansson (1970), Merton (1971,1973) and Rubinstein (1974,1976) on the necessary conditions required to attain market equilibrium, in the context of individual participant's optimal intertemporal allocation of wealth towards consumption and investment portfolios, continues to have an important influence in the analysis of most equilibrium asset pricing paradigms. The Euler first order condition arising as a result of the dynamic optimization has been used extensively as a starting point for further analysis.[1] In order to reduce the Euler condition to a more tangible form, empirical investigators have had to make specific assumptions about investor preferences. The most popular choice of utility functions has been isoelastic power utility, which is a member of the HARA class of utility functions and therefore possesses a number of desirable properties unique to this class of utility functions. In addition, isoelastic power utility implies that investor preferences are characterized by constant relative risk aversion..

Amongst the earliest attempts to obtain empirical estimates of the type of utility function typical of investor preference were two related cross-sectional studies by Friend & Blume (1975a,1975b) which examined the proportion of individual household wealth invested in risky assets as the wealth of the household increased. They concluded that the data was not inconsistent with a utility function characterized by constant relative risk aversion (CRRA); the implied CRRA coefficient being positive and larger in magnitude than unity which is required by log utility. In contrast, a study based on Canadian data by Morin & Suarez (1983) concluded that decreasing relative risk aversion was more representative. However, once the data on wealth invested in housing was reclassified for risk characteristics, in accordance with the earlier study, the findings were no longer in conflict. These studies therefore lend weight to the choice of isoelastic power utility, which exhibits CRRA.

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[1] More recent papers dealing with intertemporal equililibrium models are Lucas (1978), Breeden (1979), Prescott & Mehra (1980) and Cox, Ingersoll & Ross (1985).

Another stream of empirical research used the time series data of variables specified in various formulations of the Euler condition to obtain estimates of the CRRA coefficient.[2] Taylor series expansions were used in some studies to obtain a linearized version of the Euler condition.[3] Additional assumptions, in order to obtain a specific though more restrictive version of the Euler condition, were also used. The estimates obtained ranged from -1.4 to values in excess of +100. The wide spectrum of the estimates underscores the difficulty of obtaining accurate estimates using ex post data in a nonlinear model specification. Most of the studies relied on consumption data to obtain CRRA estimates. The time series data on consumption is comprised of estimates of the total consumer spending for the entire month. This causes temporal aggregation problems with the data, as the Euler condition requires the use of instantaneous consumption data, not monthly aggregates. Another problem relates to the treatment of consumer durable goods, since individuals derive utility from consumption of this category of goods over an extended period of time. Therefore the use of consumption data and the attendant measurement error problem further compound the complexity of the analysis.

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[2] Recent papers which have estimated CRRA include Brown & Gibbons (1985), Hansen & Singleton (1982,1983) & Litzenberger & Ronn (1986).

[3] Loistl (1976) shows that Taylor series expansion of the expected utility function do not necessarily converge, for the case of power utility. Therefore caution in using such expansions is required.

Though the nonlinearity is inherent in the model, it is possible to circumvent the problems associated with the use of consumption data. This is the approach taken by Brown & Gibbons (1985), who used the returns on the market proxy and the risk free (T-Bill) rate to obtain estimates of the CRRA coefficient. The form of the Euler equation used by them requires the additional assumption of either a constant investment opportunity set or a constant proportion of end of period wealth being reinvested, the balance being consumed. Under these conditions, Brown & Gibbons obtain estimates of CRRA which are positive with a maximum value of +7, for different time intervals used. The Method of Moments methodology used by the Brown & Gibbons study is promising, and the estimates show reasonable stability through time. However, the length of the time intervals strains the stationarity assumptions implicit in the model.[4] The study also fails to exploit fully the available security returns data by resorting to univariate estimation procedures, despite developing the multivariate model theoretically.

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[4] The method of moments estimator, though consistent, relies heavily on stationarity of the model.

This paper attempts to further the research methodology used by Brown & Gibbons (1985) by obtaining CRRA estimates using the Multivariate Nonlinear Model framework. This enables the data set to be reduced to 60 monthly observations per variable and still have available sufficient degrees of freedom, eliminating one of the drawbacks of the Brown & Gibbons (1985), (henceforth B&G), study. In addition to the Method of Moments estimator used by B&G, the Nonlinear Least Squares Estimator is also utilized to provide estimates of CRRA. This allows for comparison of the results using alternate methodologies. Because all the estimates should converge asymptotically to the true value, it is hoped that this procedure will provide some clues about the confidence with which the results of asymptotic theory can be considered valid when applied to small sample data.

The rest of the paper is organized as follows. Section I briefly describes the model specification and the estimators used. Section II looks at the results, and compares them. Concluding remarks are made in section III.

I. TEST SPECIFICATIONS FOR THE EULER CONDITION

The Euler first order condition necessary for optimal intertemporal consumption in a time additive, Von Neumann-Morgenstern expected utility maximization framework is

$$E \{ U'(C_t) (1 + R_{jt}) \mid Z^*_{t-1} \} = (1 + \Delta) U'(C_{t-1})$$

where C_t = Aggregate consumption at time 't'
R_{jt} = Stochastic rate of return for the j^{th} asset at time 't'
Δ = Rate of pure time discount
Z^*_{t-i} = Information set available at time 't-1'

Brown & Gibbons (1985) have modified the Euler condition to take on the following form :

$$E\{ (X_{jt} - 1) X_{Mt}^{-\mu} \mid Z^*_{t-1} \} = 0 \qquad (1.1)$$

where X_{jt} = $(1 + R_{jt}) / (1 + R_{Ft})$
X_{Mt} = $(1 + R_{Mt}) / (1 + R_{Ft})$
μ = Coefficient of constant relative risk aversion
Z^*_{t-i} = Information set available at time 't-1'

and

R_{jt} = Stochastic rate of return for the j^{th} asset at time 't'
R_{Mt} – Stochastic rate of return for the market proxy at time 't'
R_{Ft} = Rate of return on the risk-free asset at time 't'

Because the market portfolio also satisfies the above relationship, time series data on the returns provided by the risk free asset and the proxy for the market portfolio were used by B&G to obtain univariate nonlinear estimates of μ, the degree of aggregate constant relative risk aversion (CRRA).

Eq.(1.1) applies for markets in which investor preferences are characterised by isoelastic (power) utility. The equilibrium relationship is conditional on the information set actually used by the marketplace (i.e. full information), though the relationship is still testable for a subset of all relevant information. If the information set is deficient in any significant way, it may be reflected in the results in the form of biases or a very weak statistical relationship. Because ex post historical data is used as the information set in the actual empirical work that follows, this is a matter of some concern.

Another potential problem is the fact that the relationship applies to real returns, whereas the data set available is denominated in nominal terms. B&G address this issue by stating that a sufficient condition for the use of nominal returns is that inflation is known with certainty for each period, and is therefore part of the information set contained in Z^*_{t-1}. Though this is not an implausible assumption, an alternative assumption that the real risk free rate of interest be constant for the period under study is also sufficient. It is recognized that though the real interest rate does in fact vary, the movement is relatively small when compared with the variability of stock portfolios, and for smaller time intervals this assumption may not be very restrictive.

The following methods are used to obtain ex post estimates of CRRA :

A. *Method of Moments (MM) Estimator*

If the sample size is sufficiently large to apply the law of iterated expectations, the Method of Moments (MM) estimation technique developed by Hansen & Singleton (1982) can be used to provide an estimate of CRRA. Brown & Gibbons (1985) use this methodology in a univariate framework. They also show that for a dataset comprising N portfolios having returns vector of size T, the multivariate Method of Moments estimator is one which minimizes :

$$\text{Min } Z(\mu) = F(\mu)' \{ W^{-1} \} F(\mu) \qquad (1.2)$$

where

$F(\mu)' = (f_1(\mu), f_2(\mu), .. \quad f_j(\mu), .. \qquad f_N(\mu))$
$f_j(\mu) = \Sigma_{t=1,T} \{ (X_{jt} - 1) X_{Mt}^{-\mu} \}$
W = NxN weighing matrix with typical element
$w_{ij} = \Sigma_{t=1,T} \{ (X_{it} - 1) X_{Mt}^{-\mu} \} \{ (X_{jt} - 1) X_{Mt}^{-\mu} \}$

It can be readily seen that the Method of Moments estimation technique minimizes the weighted sum of the squared sample means of the Euler condition (1.1) applied to individual assets. Hansen & Singleton's (1982) choice of the optimal weighting matrix is designed to maximize the efficiency of the estimate. The MM procedure uses less information than the Multivariate Nonlinear Least Squares (MNLS) estimator described below as sample means rather than individual sample values are availed of in the MM estimation. Gallant (pg.153,1987) states that MM estimation has weaker consistency results when compared to MNLS. The advantage of using MM estimation is that almost no econometric assumptions (except those that guarantee consistency) are required.

B. *Multivariate Nonlinear Least Squares (MNLS) Estimator*

The Euler condition, Eq.(1.1) represents ex ante expectations based on the full information set available to the agent. The econometrician, has access to a subset of this information. As this study utilizes only ex post historical data, the empirical version of Eq.(1.1) incorporates a random error term which captures the difference between ex ante expectation and the actual ex post realization. Therefore the nonlinear regression model specification for the 'j^{th}' security is :

$$h_{jt}(\mu) = e_{jt} \tag{1.3}$$

where $h_{jt}(\mu) = (X_{jt} - 1) X_{Mt}^{-\mu}$

Eq.(1.3) is valid for all securities 'j', (j=1,N) over the sample period extending from t=1,T. The error term should have a zero mean and be uncorrelated with the time series return data, if the model specification is correct and economic agents form expectations rationally.

MNLS regression requires the residuals to be serially uncorrelated with zero means. Temporal residual correlations across equations are permitted and assumed to be the same for all time periods. The Multivariate Nonlinear Least Squares (MNLS) model is well established and details of the estimator can be obtained from Gallant (pg. 290,1987). The multivariate version of Eq.(1.3) can be specified as :

$$\mathbf{H}(\mu) = \mathbf{e} \qquad (1.3a)$$

where

$\mathbf{H}(\mu) = (\mathbf{h}_1(\mu)', \mathbf{h}_2(\mu)', ..\mathbf{h}_j(\mu)', ..\mathbf{h}_N(\mu)')'$ vector, size TNx1

$\mathbf{h}_j(\mu) = (h_{j1}, h_{j2}, .. \quad h_{jt}, .. \quad h_{jT})'$ vector of size Tx1

and

$\mathbf{e} = (\mathbf{e}_1', \mathbf{e}_2', .. \quad \mathbf{e}_j', .. \quad \mathbf{e}_N')'$ vector of size TNx1

$\mathbf{e}_j = (e_{j1}, e_{j2}, .. \quad e_{jt}, .. \quad e_{jT})'$ vector of size Tx1

with

$\mathbf{ee}' = \Sigma \otimes I_T$

where

Σ = Contemporaneous Cov Matrix of residuals , size NxN

I_T = Identity Matrix , size TxT

The MM model can only estimate the asymptotic distribution of the parameters. One of the key advantages of using the MNLS model specification, when compared with the MM estimation, is that it allows the econometrician to carry out statistical tests on parametric restrictions imposed on the model. The MNLS model as specified above can be conveniently used to test particular values of CRRA which are of interest. These are :

(a) $\mu = 1$ which implies log utility.

(b) $\mu = 0$ which implies risk neutrality.

(c) $\mu = -1$ which is consistent with quadratic power utility (and therefore CAPM).[5]

Gallant (1987,chap.5) recommends the use of the following statistic :

$$L = \frac{[\, S(\mu_r,\Sigma_u) - S(\mu_u,\Sigma_u)\,] / q}{S(\mu_r,\Sigma_u) / (NT - p)} \tag{1.3b}$$

where $S(\mu,\Sigma) = [\, H(\mu)\,]' (\Sigma^{-1} \otimes I_T)\, [\, H(\mu)\,]$

and subscripts r , u represent restricted , unrestricted estimates respectively.

Asymptotically this test statistic has an F-distribution with "q" degrees of freedom in the numerator and "(NT-p)" degrees of freedom in the denominator. Note that p=1, as the parameter space is unidimensional. Since the null hypothesis restricts this parameter, q also takes on the value of 1. In the MNLS procedure two alternate estimates of Σ may be used in the weighing scheme. The first involves making a preliminary unweighted least squares estimate of μ assuming i.i.d. residuals across equations. The residuals obtained from this preliminary estimation are used to form an estimate of the residual variance-covariance weighting matrix, Σ, which is then used in the weighted least squares estimation. The second uses the OLS residuals from separate univariate regressions to obtain an estimate of Σ, which is then incorporated as the appropriate weighting matrix in the MNLS estimation. The results obtained from the use of the two different estimates of Σ were similar. Therefore only those using the first weighting scheme are reported in section III below.

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[5] Though $\mu = -1$ is consistent with the CAPM, the model specification used in this paper implies a specific link between R_M and R_F, which is a function of aggregate market preferences represented by the value of b (see Brown and Gibbons for some discussion on this point). Since the CAPM is not a utility based asset pricing model, this constraint is not valid for the CAPM.

II. EMPIRICAL METHODOLOGY & RESULTS

As there is just one parameter to be estimated (namely μ), the algorithm to be used in the nonlinear estimation is somewhat less complex than would otherwise be the case. IMSL subroutines are used extensively. A grid search procedure was used, as the one dimensional parameter space allowed this technique to be cost effective, while at the same time allowing for full exploration of the minimization surface around the optimum value. This allowed for checks to ensure that the values obtained were in fact global minima.

A. Data Description

The univariate tests were carried out using the Ibbotson & Sinquefield data for 1 month T-Bill returns and NYSE value-weighted Index monthly returns as proxies for the risk free rate and the market index, respectively. For the multivariate tests, the data set was expanded to include the CRSP NYSE monthly stock returns tape.

The stocks were grouped into 20 portfolios. To be included a stock had to be listed on the NYSE and have at least 48 returns available in the 60 months prior to the sample period. The portfolios were formed using the Fama & MacBeth (1973) methodology, with CAPM betas as the basis for grouping. The sample size, T, is set to 60 monthly return observations (except for the results in Table 1, which replicates the B&G study and uses their sample periods). This procedure is repeated over successive 5 year intervals, starting with 1934-1938 as the first sample period.

B. Univariate Tests

Table 1 shows the results of estimation using both the Method of Moments (MM) and the Nonlinear Least Squares (NLS) estimation techniques. The estimates of CRRA vary substantially, with the method of moments estimates showing consistently higher values. The sample periods are identical to those in the B&G study and the MM estimates are close to those obtained by B&G. The MM estimates range from a minimum of 0.360 to a maximum of 7.126, with NLS providing estimates ranging between -0.422 and 1.695. This suggests that the B&G estimates may be biased upwards, if the NLS estimator is less susceptible to small sample bias. This is quite possible because as mentioned earlier, Gallant (1987) states that the MM estimator has weaker consistency properties when compared with the MNLS estimator. Only a simulation study can ascertain the actual small sample bias present in the two estimators, and this issue is a topic for future research.

The restriction implied by $\mu = -1$ (i.e. consistent with CAPM) and $\mu = 1$ (i.e. consistent with LOG utility) specifications is rejected at the 95% significance level for most sample periods, while the risk NEUTRAL specification (μ=0) is accepted most of the time. The standard errors for the estimates of μ using NLS methodology are quite small as can be seen by the size of the 95% C.I. (which is approximately four times the standard error, assuming a normal distribution).

Table 2 carries out the CRRA estimation using the same univariate methodology as that in table 1 with the sample period contracted to 60 months. As would be expected, the smaller sample size provides estimates which are spread over a wider range (-1.617 to 12.847 for MM; -2.908 to 6.957 for NLS), with the 95% C.I. larger than before. As a consequence of this, the restriction implied by the quadratic and risk neutral utility (CRRA values -1 & 0 resp.) is rejected 3 times each, while log utility (CRRA = 1) is rejected 4 times, in the 10 sample periods studied. Therefore this model specification, when used for a sample of size 60, lacks sufficient power to discriminate between the Log, Risk neutral and CAPM consistent values of CRRA. A judicious choice of sample period which optimizes between the conflicting requirements of larger sample size and stationarity of the system is therefore indicated, when univariate methodology is used. The approach taken by this study is to resort to the potentially more powerful multivariate regression procedures, including portfolio returns in the expanded data set. This inclusion of portfolios returns also implies a fuller use of the available information set.

C. Multivariate Tests

Table 3 tabulates CRRA estimates obtained using 20 portfolios and the multivariate methodology outlined in section II. The sample periods used were identical to those in table 2. As expected, the 95% C.I. for the MNLS estimates are now considerably smaller, reflecting the use of a broader data set. A direct consequence of this is the failure to accept the restricted values of CRRA as much as nine times out of ten for the restrictions consistent with log utility or quadratic utility, while risk neutral utility is not accepted for all ten sample periods. A surprising feature of the multivariate estimates is the fact that the range of values is broader than that obtained using univariate methodology. The MM estimates range from -2.99 to 17.60, while the MNLS estimates give values ranging from -1.56 to 9.70. This indicates that there is a possibility that the estimation errors present in the univariate techniques is accentuated in the multivariate methodology. Once again simulation would be required to test this conjecture. The multivariate results confirm the earlier finding that the MM estimates are higher than the MNLS estimate; except in the one case when the MM estimate is negative (1969-73).

D. Discussion of Results

The results indicate potential for instability and error in the estimates of CRRA, regardless of the particular methodology used. The extent of estimation error is harder to pin down. An analysis of the results seems to indicate that there is a larger possibility of bias in the case of the MM estimator, when compared with the NLS estimator. The fact that the NLS estimator penalizes individual residuals as a squared value, it being a function of the sum of the squared errors, may explain this result. By its very specification, the NLS estimator penalizes heavily residuals which are large in magnitude. This causes the NLS estimator to be more sensitive to parameter shifts that create larger outliers. In contrast, the MM estimator is based on the minimization of the squared mean error. Therefore it takes into consideration only shifts in the residual mean. This makes it less sensitive to the dispersion of the individual residuals, as only the residual mean is part of the objective function in the minimization. Therefore when compared with the NLS estimator, it possesses a greater tolerance for individual outliers and skewness in the residual distribution. This allows the objective function algorithm to search farther afield in its attempt to reach a minimum. The estimates for 1969-73 seem to lend credence to this view. This is the only time period in which both methods provide negative estimates of CRRA. The MM estimate is algebraically smaller but larger in magnitude. In all other time intervals, with the exception of the 1974-78 period, the MM is both positive and larger in magnitude. However, as mentioned earlier, a simulation study is required to move the above analysis from conjecture to a more substantive footing.

There is one other point that needs discussion. Nonlinear models explicitly rely on asymptotic theory. Therefore it is very difficult to make rigorous comparisons with other competing linear models. However, it should be pointed out that whereas the well known CAPM tends to reduce the residual variance of grouped portfolio returns, conditional on the market return by about 20% for similar data, the model used in this paper has a very low explanatory power. In fact, the conditional variance of asset returns is almost unchanged from the unconditional variance. This is despite the sensitivity exhibited to shifts in the parameter values of CRRA. Therefore the use of this model to obtain meaningful estimates of CRRA is put in question. A possible reason for this inability to explain return movements may be the fact that a market proxy is being used. The nonlinear way the market index interacts with the CRRA parameter may accentuate this error-in-variables problem, raising its severity when compared to the CAPM. Other causes could be the use of ex post data and a limited information set.

III. CONCLUSION

Comparison of the estimates of CRRA obtained by the two methods which produced usable estimates shows that the possibility of significant small sample bias exists. The extent of bias can be obtained only from a simulation study, but the results obtained tend to suggest that there is potentially more bias present in the case of the MM estimator, when compared with the MNLS estimator. Whether this is valid more generally or applies only in the case of this specific study requires a more thorough investigation.

The model specification used in the study has low explanatory power, with the conditional variance of asset returns almost unchanged from the unconditional variance. The estimates themselves show quite a bit of variability through time, and indicate that the assumption of CRRA stationarity through time is not born out by the results, especially in the multivariate case. The estimates seem plausible for most time periods with the obvious caveats stemming from the use of ex post data.

Table 1
Univariate Estimates of Constant Relative Risk Aversion

Univariate estimates of constant relative risk aversion (CRRA) are reproduced below using both the Method of Moments (MM) and Nonlinear Least Squares methodology. The sample periods used are the same as that by Brown & Gibbons (1985). The MM estimates conform closely to their results. The confidence intervals are obtained by inverting the F test statistic.

	MM	NONLINEAR LEAST SQUARES					
PERIOD	Estimate	Estimate	95%	C.I	F TEST STATISTIC		
	μ	μ	max	min	CAPM	LOG	NEUTRAL
1/26-12/81	1.860	0.389	0.69	0.09	82.15	15.76	6.47
1/26-12/52	1.542	0.391	0.77	0.01	50.26	9.65	4.04
1/53-12/81	2.784	0.346	1.29	-0.59	7.84	1.86*	0.52*
1/26- 6/39	0.631	0.452	0.96	-0.05	31.26	4.53	3.09*
7/39-12/52	4.726	-0.422	0.52	-1.45	1.26*	8.97	0.73*
1/53- 6/67	7.126	1.695	3.44	-0.05	9.03	0.61*	3.61*
7/67-12/81	0.360	-0.011	1.19	-1.20	2.64*	2.74*	0.00*

Note: 1. CAPM -> Ho: $\mu=-1$; LOG -> Ho: $\mu=1$; NEUTRAL -> Ho: $\mu=0$.

2. Critical value at 95% significance level is 3.84.

3. * indicates Ho not rejected.

Table 2
Univariate Estimates of CRRA using 60 month sample size

The results in this table utilize the same methodology as in Table 1. The sample size, however, is restricted to 60 observations.

	MM	NONLINEAR LEAST SQUARES					
PERIOD	Estimate μ	Estimate μ	95% C.I max	min	F TEST STATISTIC CAPM	LOG	NEUTRAL
1979-83	3.069	1.036	3.2	-1.1	3.23	0.00	0.85
1974-78	0.248	1.150	2.9	-0.6	5.52**	0.03	1.58
1969-73	-1.617	-2.908	-0.4	-5.6	2.12	9.07**	4.98**
1964-68	6.446	1.602	4.9	-1.9	2.57	0.14	0.98
1959-63	4.294	0.202	2.8	-2.4	0.79	0.35	0.02
1954-58	12.126	5.933	9.5	2.5	14.62**	7.56**	10.83**
1949-53	12.847	6.957	10.9	3.0	14.57**	8.41**	11.31**
1944-48	4.977	0.865	3.4	-1.9	1.83	0.01	0.40
1939-43	1.618	-1.102	0.2	-2.5	0.02	10.10**	2.66
1934-38	2.086	-0.014	1.0	-1.1	2.97	3.41	0.00

Note: 1. CAPM -> Ho: $\mu=-1$; LOG -> Ho: $\mu=1$; NEUTRAL -> Ho: $\mu=0$.

2. Critical value for F(1, 60) at 95% significance is 3.84.

3. ** indicates Ho rejected.

Table 3
MM & NLS Estimation of RRA using 20 portfolios

This table lists estimates of Constant Relative Risk Aversion obtained when multivariate methodology is employed. 20 portfolios were used for the identical 60 month sample periods specified in Table 2.

	RELATIVE RISK AVERSION ESTIMATES				F TEST STATISTIC		
PERIOD	MM	Nonlinear Least Squares Estimate	95% C.I. max.	min.	CAPM	LOG	NEUTRAL
1979-83	6.064	2.703	3.53	1.87	77.47	16.19	40.93
1974-78	2.314	2.327	3.00	1.66	104.73	15.61	49.39
1969-73	-2.992	-1.560	-0.68	-2.46	1.54*	33.98	12.27
1964-68	12.355	4.404	5.63	3.17	75.11	29.43	49.51
959-63	5.379	3.001	4.01	1.98	57.72	14.57	32.57
1954-58	17.602	9.453	10.53	8.38	402.87	253.73	323.00
1949-53	15.004	9.700	11.00	8.40	280.53	180.78	227.47
1944-48	5.822	3.347	4.17	2.51	99.83	29.45	59.44
1939-43	2.962	1.145	5.78	1.69	51.83	0.26*	15.32
1934-38	2.577	2.152	2.63	1.66	148.60	20.72	70.14

Note: 1. CAPM -> Ho: $\mu=-1$; LOG -> Ho: $\mu=1$; NEUTRAL -> Ho: $\mu=0$.

2. Critical value for F(1, 1119) at 95% significance is 3.84.

3. * indicates Ho not rejected.

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